THERMO-ELASTICITY FOR ANISOTROPIC MEDIA IN HIGHER DIMENSIONS

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Abstract. In this note we develop tools to study the Cauchy problem for the system of thermo-elasticity in higher dimensions. The theory is developed for general homogeneous anisotropic media under non-degeneracy conditions.

For degenerate cases a method of treatment is sketched and for the cases of cubic media and hexagonal media detailed studies are provided.

1. Introduction

While isotropic thermo-elasticity is a well-known and well-established subject (see, e.g., the book of Jiang Song-Racke [15] and references therein) only very few results are available for the case of anisotropic media. Among them are the theses of Borkenstein [1] for cubic media and Doll [3] for the case of rhombic media together with the authors treatments [16], [22], all in two space dimensions.

In this paper the system of anisotropic thermo-elasticity in three (and more) dimensions, i.e.,

$$U_{tt} + A(\mathrm{D})U + \gamma\nabla\theta = 0, \tag{1.1a}$$

$$\theta_t - \kappa\Delta\theta + \gamma\nabla\cdot U_t = 0 \tag{1.1b}$$

for the elastic displacement $U(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and temperature difference $\theta(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ to the equilibrium state, will be considered. The system (1.1) couples the hyperbolic elasticity equation with the parabolic heat equation. The operator $A(\mathrm{D})$ describes the elastic properties of the underlying medium, while κ denotes its thermal conductibility. The constant γ describes the thermo-elastic coupling. Basic assumptions of our theory are $\kappa > 0$, $\gamma^2 > 0$ together with

- $A(\xi) = |\xi|^2 A(\eta)$, $\eta = \xi/|\xi|$, is a 2-homogeneous matrix-valued symbol;
- $A : \mathbb{S}^{n-1} \to \mathbb{R}^{n\times n}$ is a real-analytic function of $\eta \in \mathbb{S}^{n-1}$, $n \geq 3$;
- $A(\eta) = A^*(\eta) > 0$ is self-adjoint and positive.

In general we can *not* assume that $A(\eta)$ is non-degenerate in the sense that $\#\operatorname{spec} A(\eta) = n$ for all $\eta \in \mathbb{S}^{n-1}$ (as done for the two-dimensional case in [16]). All basic examples show degeneracies in dimensions $n \geq 3$.

Example 1.1. *Isotropic media*

$$A(\eta) = \mu I + (\lambda + \mu)\eta \otimes \eta \tag{1.2}$$

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with Lamé constants λ and μ. The matrix $A(\eta)$ is positive as long as $\mu > 0$ and $\lambda > -2\mu$. The eigenvectors of $A(\eta)$ are multiples of η and $\eta^\perp$ and thus invariant under rotations of frequency space.

Example 1.2. *Cubic media*

$$A(\eta) = \begin{pmatrix} (\tau-\mu)\eta_1^2+\mu & (\lambda+\mu)\eta_1\eta_2 & \cdots & (\lambda+\mu)\eta_1\eta_n \\ (\lambda+\mu)\eta_1\eta_2 & (\tau-\mu)\eta_2^2+\mu & & \vdots \\ \vdots & & \ddots & \vdots \\ (\lambda+\mu)\eta_1\eta_n & \cdots & \cdots & (\tau-\mu)\eta_n^2+\mu \end{pmatrix} \tag{1.3}$$

described by parameters λ, μ and τ. Later we will describe the assumptions made on these parameters and the resulting spectral properties of the matrix function $A(\eta)$ more precise. In the case of three space dimensions, the matrix $A(\eta)$ is positive if and only if $\mu > 0$, $\tau > 0$ together with $-2\mu - \tau/2 < \lambda < \tau$. In three space dimensions this will be one of our main examples.

Example 1.3. We can replace the constant τ on the diagonal by $\tau_1, \dots, \tau_n$ in (1.3). This yields so-called *rhombic media*. The behaviour of rhombic media is close to that of cubic media if the parameters are of similar size, in general there will appear exceptional situations. See, e.g., [22] or [24] for a discussion of this effect in two space dimensions.

Example 1.4. *Hexagonal media* are another particularly interesting case for three space dimensions. Since we want to come back to them later on we introduce the corresponding operator. It is given by

$$A(\eta) = \mathbb{D}(\eta)^T \mathcal{C} \mathbb{D}(\eta), \tag{1.4}$$

where $\mathcal{C}$ contains the 5 structure constants τ_1, τ_2, λ_1, λ_2 and μ and $\mathbb{D}(\eta)$ is of a particular form,

$$\mathcal{C} = \begin{pmatrix} \tau_1 & \lambda_1 & \lambda_2 & & & \\ \lambda_1 & \tau_1 & \lambda_2 & & & \\ \lambda_2 & \lambda_2 & \tau_2 & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \frac{\tau_1-\lambda_1}{2} \end{pmatrix}, \qquad \mathbb{D}(\eta) = \begin{pmatrix} \eta_1 & & \\ & \eta_2 & \\ & & \eta_3 \\ & \eta_3 & \eta_2 \\ \eta_3 & & \eta_1 \\ \eta_2 & \eta_1 & \end{pmatrix}. \tag{1.5}$$

Even the first (non-trivial anisotropic) example, the case of cubic media in three space dimensions, has degenerate directions in which $A(\eta)$ has double eigenvalues. Later on we will analyse this example in detail.

Definition 1. We call a direction $\eta \in \mathbb{S}^{n-1}$ (elastically) *non-degenerate* if

$$\#\operatorname{spec} A(\eta) = n \tag{1.6}$$

holds true for this direction η.

The set of non-degenerate directions is an open subset of $\mathbb{S}^{n-1}$. For non-degenerate directions the treatment of [16] transfers almost immediately and gives a representation of solutions. We will sketch the results in Section 2. In Section 3 we consider special degenerate directions and discuss the examples of cubic and hexagonal media. Dispersive estimates for solutions are given in Section 4. In the neighbourhood of degenerate

directions they are essentially based on estimates developed by Liess [8], [10] for the treatment of anisotropic acoustic equations.

2. Treatment of non-degenerate directions

For the following we consider a simply connected open subset $\mathcal{U}$ of $\mathbb{S}^{n-1}$, where the symbol $A(\eta)$ has n distinct (and real) eigenvalues. We denote these eigenvalues in ascending order as

$$0 < \varkappa_1(\eta) < \varkappa_2(\eta) < \cdots < \varkappa_n(\eta). \tag{2.1}$$

By analytic perturbation theory, see [6], we know that these eigenvalues are real-analytic and that we find corresponding normalised eigenvectors

$$r_1(\eta), \ldots, r_n(\eta) \in C^\infty(\mathcal{U}, \mathbb{S}^{n-1}) \tag{2.2}$$

depending analytically on $\eta \in \mathcal{U}$. Collecting them in the unitary matrix

$$M(\eta) = \big(r_1(\eta)|r_2(\eta)|\cdots|r_n(\eta)\big), \tag{2.3}$$

$$M^*(\eta)M(\eta) = I = M(\eta)M^*(\eta), \tag{2.4}$$

we can diagonalise the matrix $A(\eta)$

$$A(\eta)M(\eta) = M(\eta)\mathcal{D}(\eta), \tag{2.5}$$

$$\mathcal{D}(\eta) = \operatorname{diag}\big(\varkappa_1(\eta), \varkappa_2(\eta), \ldots, \varkappa_n(\eta)\big). \tag{2.6}$$

In our treatment we will not make use of analyticity directly, instead our use of perturbation theory will be based on [5] und [23] and uses only smooth dependence. This will be of interest for generalisations later on. Therefore, whenever we use analyticity, we will explicitly state that.

We use $M(\eta)$ to reduce the thermo-elastic system to a system of first order. For this we denote by $\hat{U}$ and $\hat{\theta}$ the partial Fourier transforms of U and θ with respect to the spatial variables and consider

$$V = \begin{pmatrix} (\mathrm{D}_t + \mathcal{D}^{1/2}(\xi))M^*(\eta)\hat{U} \\ (\mathrm{D}_t - \mathcal{D}^{1/2}(\xi))M^*(\eta)\hat{U} \\ \hat{\theta} \end{pmatrix} \in \mathbb{C}^{2n+1}, \tag{2.7}$$

as usual $\mathrm{D}_t = -\mathrm{i}\partial_t$ and $\eta = \xi/|\xi|$. Then V satisfies a first order system of ordinary differential equations, which has an apparantly simple structure. Straightforward calculation shows that

$$\mathrm{D}_t V = B(\xi)V \tag{2.8}$$

holds true with coefficient matrix

$$B(t,\xi) = \begin{pmatrix} \omega_1(\xi) & & & & & & \mathrm{i}\gamma a_1(\xi) \\ & \omega_2(\xi) & & & & & \mathrm{i}\gamma a_2(\xi) \\ & & \ddots & & & & \vdots \\ & & & -\omega_1(\xi) & & & \mathrm{i}\gamma a_1(\xi) \\ & & & & -\omega_2(\xi) & & \mathrm{i}\gamma a_2(\xi) \\ & & & & & \ddots & \vdots \\ \frac{\mathrm{i}\gamma}{2}a_1(\xi) & \frac{\mathrm{i}\gamma}{2}a_2(\xi) & \cdots & \frac{\mathrm{i}\gamma}{2}a_1(\xi) & \frac{\mathrm{i}\gamma}{2}a_2(\xi) & \cdots & \mathrm{i}\kappa|\xi|^2 \end{pmatrix}, \tag{2.9}$$

where $\omega_j(\xi) = \sqrt{\varkappa_j(\xi)} \in C^\infty(\mathcal{U}, \mathbb{R}_+)$ and
$$a_j(\xi) = r_j(\eta) \cdot \xi. \tag{2.10}$$
Following the conventions of [16] we denote these functions $a_j(\xi)$ as the *coupling functions* of the thermo-elastic system associated to the elastic operator $A(\mathrm{D})$. They play a prominent rôle for the description of the time-asymptotic behaviour of solutions. This reflects the fact that they couple the homogeneous first order entries in $B(\xi)$ with the second order lower right corner entry. Note, that
$$\sum_{j=1}^n a_j^2(\eta) = 1. \tag{2.11}$$
Zeros of the coupling functions are of particular importance. Following [16, Def. 1] we define:

Definition 2. A non-degenerate direction $\eta \in \mathbb{S}^{n-1}$ is called

- *hyperbolic* if one of the coupling functions vanishes; more precisely, it is called *hyperbolic with respect to the eigenvalue* $\varkappa_j(\eta)$ if $a_j(\eta) = 0$;
- *parabolic* if all coupling functions are non-zero.

In the anisotropic case the set of hyperbolic directions is (generically[1]) a lower dimensional subset of $\mathbb{S}^{n-1}$. In order to decide whether a direction is hyperbolic or parabolic we can employ the following proposition. We denote for a matrix A and a vector η by
$$\mathcal{Z}(A,\eta) = \operatorname{span}\{\, A^k\eta \,|\, k = 0, 1, \dots \} \tag{2.12}$$
the corresponding cyclic subspace, i.e. the span of the trajectory of η under the action of the matrix A.

Proposition 2.1. *The following statements are equivalent:*

(1) *The cyclic subspace of η has dimension $n-k$, i.e., $\dim \mathcal{Z}(A(\eta),\eta) = n-k$.*
(2) *Exactly k of the coupling functions vanish in η.*

Hence, a non-degenerate direction $\eta \in \mathbb{S}^{n-1}$ is parabolic if and only if $\mathcal{Z}(A(\eta),\eta) = \mathbb{R}^n$ and therefore
$$\det\left(\eta | A(\eta)\eta | \cdots | A^{n-1}(\eta)\eta\right) \neq 0. \tag{2.13}$$

Proof. If we represent η in the eigenbasis of $A(\eta)$ we obtain
$$\eta = a_1(\eta) r_1(\eta) + \cdots + a_n(\eta) r_n(\eta) \tag{2.14}$$
and therefore
$$A^\ell(\eta)\eta = \varkappa_1^\ell(\eta) a_1(\eta) r_1(\eta) + \cdots + \varkappa_n^\ell(\eta) a_n(\eta) r_n(\eta). \tag{2.15}$$
If k of the coupling functions vanish, then $A^{n-k}(\eta)\eta$ must be in the span of the $A^\ell(\eta)\eta$ with $\ell = 0, 1, \dots, n-k-1$ and thus the cyclic subspace is at most of dimension $n-k$. On the other hand, the first $n-k$ vectors in the trajectory are linearly independent since the corresponding matrix in the basis representation with respect to

[1] If not, by analyticity it follows that one coupling function vanishes on $\mathcal{U}$ and the system is therefore decoupled. This case is reduced to the study of the lower dimensional blocks, one is a hyperbolic system the other one a thermo-elastic system of lower dimension. This is, e.g., the case for hexagonal media, see Section 3.4.

$a_1(\eta)r_1(\eta),\dots,a_n(\eta)r_n(\eta)$ is just the van der Monde matrix associated to the eigenvalues of $A(\eta)$ for non-vanishing coupling functions and therefore regular. □

2.1. **On the characteristic polynomial of the full symbol.** At first we collect some of the spectral properties of the matrix $B(\xi)$ which are directly related to the characteristic polynomial of $B(\xi)$.

Proposition 2.2. *The following identies hold true:*

$$\operatorname{tr} B(\xi) = \mathrm{i}\kappa|\xi|^2, \tag{2.16}$$

$$\det B(\xi) = \mathrm{i}\kappa|\xi|^2 \det A(\xi), \tag{2.17}$$

$$\det(\nu - B(\xi)) = (\nu - \mathrm{i}\kappa|\xi|^2)\prod_{j=1}^n(\nu^2 - \varkappa_j(\xi)) - \nu\gamma^2\sum_{j=1}^n a_j^2(\xi)\prod_{k\neq j}(\nu^2 - \varkappa_k(\xi)). \tag{2.18}$$

Furthermore, the matrix $B(\xi)$ has a purely real eigenvalue for $\xi \neq 0$ if and only if the direction $\eta = \xi/|\xi|$ is hyperbolic. If it is j-hyperbolic, then $\pm\omega_j(\xi) \in \operatorname{spec} B(\xi)$.

The proof of the last fact is fairly straightforward and consists of separating real and imaginary parts of the characteristic polynomial. Note that for all parabolic directions we can divide the characteristic polynomial by $\nu\prod_j(\nu^2 - \varkappa_j(\xi))$ to obtain

$$1 = \frac{\mathrm{i}\kappa|\xi|^2}{\nu} + \gamma^2\sum_{j=1}^n \frac{a_j^2(\xi)}{\nu^2 - \varkappa_j(\xi)}. \tag{2.19}$$

This formulation allows to consider the neighbourhoods of hyperbolic directions. Assume for this that the set of hyperbolic directions with respect to $\varkappa_j(\eta)$

$$M_j = \{\eta \in \mathcal{U} \mid a_j(\eta) = r_j(\eta)\cdot\eta = 0\} \tag{2.20}$$

is a regular submanifold of $\mathcal{U}$. If we consider the corresponding *hyperbolic* eigenvalues $\nu_j^\pm(\xi)$ of $B(\xi)$ in a neighbourhood of M_j, i.e. the eigenvalues which satisfy

$$\lim_{\eta\to M_j} \nu_j^\pm(\xi) = \pm\omega_j(\xi) \tag{2.21}$$

for fixed $|\xi|$, equation (2.19) gives a precise description of the behaviour of the imaginary part of these eigenvalues. The proof is a straightforward generalisation from [16, Prop. 2.2].

Proposition 2.3. *The non-tangential limit*

$$\lim_{\eta\to M_j} \frac{a_j^2(\xi)}{\nu_j^\pm(\xi)^2 - \varkappa_j(\xi)} = 1 \mp \frac{\mathrm{i}\kappa|\xi|^2}{\omega_j(\xi)} - \gamma^2\sum_{k\neq j}\frac{a_k^2(\xi)}{\varkappa_j(\xi) - \varkappa_k(\xi)} = \gamma^2(C_{\bar\eta} \mp \mathrm{i}D_{\bar\eta}|\xi|) \tag{2.22}$$

exists and is non-zero for all $\xi \neq 0$. Furthermore,

$$\lim_{\eta\to M_j}\frac{\operatorname{Im}\nu_j^\pm(\xi)}{a_j^2(\eta)} = \frac{D_{\bar\eta}|\xi|^2}{2\omega_j(\bar\eta)(C_{\bar\eta}^2 + |\xi|^2\mathrm{D}_{\bar\eta}^2)} > 0. \tag{2.23}$$

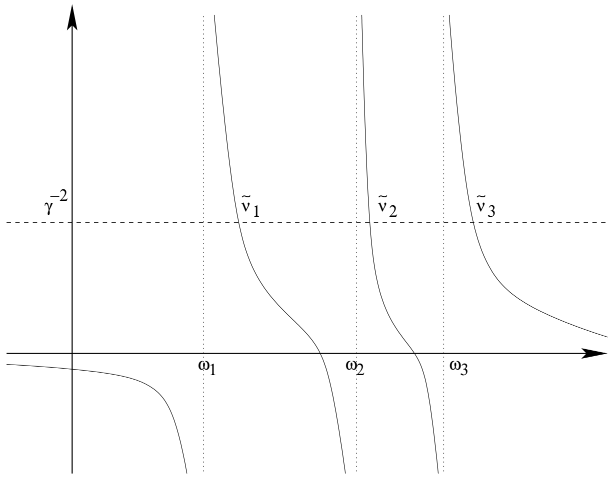

FIGURE 1. Non-zero eigenvalues of $B_1(\xi)$ for parabolic directions.

2.2. **Asymptotic expansion of the eigenvalues as $|\xi| \to 0$.** We decompose $B(\xi)$ into homogeneous components $B(\xi) = B_1(\xi) + B_2(\xi)$ of degree 1 and 2, respectively. For sufficiently small $|\xi|$ we expect the eigenvalues of $B(\xi)$ to be close to the eigenvalues of $B_1(\xi)$. For parabolic directions the (non-zero) eigenvalues of $B_1(\eta)$ can be determined from the equation

$$\frac{1}{\gamma^2} = \sum_{j=1}^{n} \frac{a_j^2(\eta)}{\tilde{\nu}^2 - \varkappa_j(\eta)}, \tag{2.24}$$

which follows directly from (2.19) with $\kappa = 0$. It can be solved (e.g. graphically, see Figure 1 for $n = 3$) to obtain the distinct eigenvalues $0, \pm\tilde{\nu}_1(\eta), \ldots, \pm\tilde{\nu}_n(\eta)$ ordered as

$$0 < \omega_1(\eta) < \tilde{\nu}_1(\eta) < \omega_2(\eta) < \tilde{\nu}_2(\eta) < \cdots < \omega_n(\eta) < \tilde{\nu}_n(\eta). \tag{2.25}$$

For hyperbolic directions a similar result holds true. In the case of hyperbolic directions w.r.to $\varkappa_j(\eta)$ eigenvalues move to $\omega_j(\eta)$. According to the choice of the coupling constant γ different cases occur:

(1) if $\frac{1}{\gamma^2}$ is large then $\tilde{\nu}_j(\eta) = \omega_j(\eta)$, the other inequalities are unchanged;
(2) if $\frac{1}{\gamma^2}$ is small then $\tilde{\nu}_{j-1}(\eta) = \omega_j(\eta)$ and the other inequalities remain true.

The critical threshold between these two cases is

$$\frac{1}{\gamma^2} = \sum_{k \neq j} \frac{a_k^2(\eta)}{\varkappa_j(\eta) - \varkappa_k(\eta)}, \tag{2.26}$$

where $B_1(\eta)$ has the *double* eigenvalue $\tilde{\nu}_{j-1}(\eta) = \omega_j(\eta) = \tilde{\nu}_j(\eta)$. Following the conventions from [16] we define:

Definition 3. We denote a hyperbolic direction w.r.to $\varkappa_j(\eta)$ as γ-degenerate if (2.26) holds true.

For the following treatment we exclude γ-degenerate hyperbolic directions and assume instead that for all hyperbolic directions in $\mathcal{U}$ condition (2.26) is not satisfied for the corresponding index j. Then the following statement is apparent.

Proposition 2.4. *Let η be not γ-degenerate. Then the matrix $B_1(\eta)$ has $2n+1$ distinct real eigenvalues $0, \pm\tilde\nu_1, \dots, \pm\tilde\nu_n$ for all $\eta \in \mathcal{U}$.*

Proposition 2.4 allows to apply the standard diagonalisation scheme (see [5, Sec. 2.1]) to $B(\xi) = B_1(\xi) + B_2(\xi)$ as $\xi \to 0$. Hence, eigenvalues, eigenprojections and all their derivatives have full asymptotic expansions as $\xi \to 0$. The proof is almost identical to that from [16, Prop. 2.5] and is omitted.

Proposition 2.5. *For all not γ-degenerate directions $\eta = \xi/|\xi| \in \mathcal{U}$ the eigenvalues and eigenprojections of $B(\xi)$ have full asymptotic expansions as $\xi \to 0$. The main terms are given by*

$$\nu_0(\xi) = \mathrm{i}\kappa|\xi|^2 b_0(\eta) + \mathcal{O}(|\xi|^3) \tag{2.27a}$$

$$\nu_j^\pm(\xi) = \pm|\xi|\tilde\nu_j(\eta) + \mathrm{i}\kappa|\xi|^2 b_j(\eta) + \mathcal{O}(|\xi|^3) \tag{2.27b}$$

with

$$b_0(\eta) = \left(1 + \gamma^2 \sum_{k=1}^n \frac{a_k^2(\eta)}{\varkappa_k(\eta)}\right)^{-1} > 0 \tag{2.28a}$$

and

$$b_j(\eta) = \left(1 + \gamma^2 \sum_{k=1}^n a_k^2(\eta) \frac{\tilde\nu_j^2(\eta) + \varkappa_k(\eta)}{(\tilde\nu_j^2(\eta) - \varkappa_k(\eta))^2}\right)^{-1} \geq 0. \tag{2.28b}$$

Furthermore, $b_j(\eta) = 0$ if and only if η is hyperbolic with respect to the eigenvalue $\varkappa_j(\eta)$.

Remark 2.1. Note, that $\operatorname{tr} B(\xi) = \mathrm{i}\kappa|\xi|^2$ implies

$$b_0(\eta) + 2\sum_{j=1}^n b_j(\eta) = 1. \tag{2.29}$$

Recall that by Proposition 2.2 eigenvalues of $B(\xi)$ can only be real along hyperbolic directions (and then they are exactly the 'trivial' real eigenvalues). In combination with the fact that eigenvalues of $B(\xi)$ are continuous in ξ we obtain:

Corollary 2.6. *For all parabolic directions $\eta = \xi/|\xi| \in \mathcal{U}$ we have $\operatorname{Im} \nu_j^\pm(\xi) > 0$. The same is true as long as η is* not *hyperbolic w.r.to $\varkappa_j(\eta)$.*

2.3. **Asymptotic expansion of the eigenvalues as $|\xi| \to \infty$.** In this case the two-step procedure developed in [5, Sec. 2.2], [16, Prop. 2.6] applies in analogy. Essential assumption is the non-degeneracy of $A(\eta)$. We omit the proof and cite the corresponding result only.

Proposition 2.7. *For all non-degenerate directions the eigenvalues and eigenprojections of the matrix $B(\xi)$ have full asymptotic expansions as $|\xi| \to \infty$. The first terms are given by*

$$\nu_0(\xi) = \mathrm{i}\kappa|\xi|^2 - \frac{\mathrm{i}\gamma}{\kappa} + \mathcal{O}(|\xi|^{-1}), \tag{2.30a}$$

$$\nu_j^\pm(\xi) = \pm|\xi|\omega_j(\eta) + \frac{\mathrm{i}\gamma^2}{2\kappa} a_j^2(\eta) + \mathcal{O}(|\xi|^{-1}). \tag{2.30b}$$

Remark 2.2. Despite the fact that we used the same notation for the eigenvalues as $\xi \to 0$ and $|\xi| \to \infty$, we do not claim that they are indeed the same functions of ξ. This is only true for hyperbolic eigenvalues near hyperbolic directions, in general there might be multiplicities in between and there might be no consistent notation for these functions.

Corollary 2.8. *For all parabolic directions $\eta = \xi/|\xi|$ the eigenvalues of $B(\xi)$ satisfy $\operatorname{Im}\nu(\eta) \geq C_\eta > 0$ for $|\xi| \geq c$. The same is true for parabolic eigenvalues in hyperbolic directions.*

Remark 2.3. In particular, we see by the asymptotic expansions that the eigenvalues of $B(\xi)$ are simple for large and also for small values of $|\xi|$. Furthermore, we see that the hyperbolic eigenvalues are always separeted (i.e. if multiplicities occur in hyperbolic directions, they involve only parabolic eigenvalues).

2.4. **Behaviour of the imaginary part.** The asymptotic expansions of Propositions 2.5 and 2.7 allow to draw conclusions for the behaviour of the imaginary part. We collect them for later use. The first result is apparent.

Proposition 2.9. *On any compact set of parabolic directions we have the uniform estimates*

$$\operatorname{Im}\nu_j^{(\pm)}(\xi) \geq C_\epsilon \qquad \text{for all } |\xi| \geq \epsilon, \tag{2.31}$$

$$\operatorname{Im}\nu_j^{(\pm)}(\xi) \sim b_j(\eta)|\xi|^2 \qquad \text{for all } |\xi| \leq \epsilon \tag{2.32}$$

for all eigenvalues of $B(\xi)$ and arbitrary $\epsilon > 0$.

The next statement is concerned with a tubular neighbourhood of a compact subset of a regular submanifold M_j of hyperbolic eigenvalues w.r.to $\varkappa_j(\eta)$. It is only of interest how the corresponding hyperbolic eigenvalues $\nu_j^\pm(\xi)$ behave, the others still satisfy Proposition 2.9.

Proposition 2.10. *Uniformly on any tubular neighbourhood of a compact subset of M_j of non-γ-degenerate directions the corresponding hyperbolic eigenvalues $\nu_j^\pm(\xi)$ satisfy the estimates*

$$\operatorname{Im}\nu_j^{\pm}(\xi) \sim a_j^2(\eta) \qquad \text{for all } |\xi| \geq \epsilon, \tag{2.33}$$

$$\operatorname{Im}\nu_j(\xi) \sim b_j(\eta)|\xi|^2 \qquad \text{for all } |\xi| \leq \epsilon. \tag{2.34}$$

Proof. By Proposition 2.3 we know that

$$\operatorname{Im}\nu_j^{\pm}(\xi) = a_j^2(\xi)K(\xi) \tag{2.35}$$

for some function $K(\xi)$. Our aim is to estimate $K(\xi)$. The left hand of this formula has a full asymptotic expansion as $|\xi| \to 0$ and $|\xi| \to \infty$. Therefore, also the right hand side has one and it follows that

$$K(\xi) = \frac{\gamma^2}{2\kappa} + \mathcal{O}(|\xi|^{-1}), \qquad |\xi| \to \infty, \tag{2.36a}$$

$$K(\xi) = \kappa|\xi|^2 \frac{b_j(\eta)}{a_j^2(\eta)} + \mathcal{O}(|\xi|^{-3}), \qquad |\xi| \to 0. \tag{2.36b}$$

Thus, the desired estimate follows by a compactness argument as soon as we have a uniform lower/upper bound for $b_j(\eta)/a_j^2(\eta)$. The representation of $b_j(\eta)$ in Proposition 2.5 in combination with (2.24) implies

$$\begin{aligned}\lim_{\eta\to M_j}\frac{a_j^2(\eta)}{b_j(\eta)} &= \lim_{\eta\to M_j}\gamma^2(\tilde\nu_j^2+\varkappa_j(\eta))\frac{a_j^4(\eta)}{(\tilde\nu_j^2-\varkappa_j(\eta))^2}\\ &\quad+\lim_{\eta\to M_j}a_j^2(\eta)\left(1+\gamma^2\sum_{k\neq j}a_k^2(\eta)\frac{\tilde\nu_j^2+\varkappa_k(\eta)}{(\tilde\nu_j^2-\varkappa_k(\eta))^2}\right)\\ &=2\gamma^2\varkappa_j(\bar\eta)\left(1-\gamma^2\sum_{j\neq k}\frac{a_k^2(\bar\eta)}{\varkappa_j(\bar\eta)-\varkappa_k(\bar\eta)}\right)^2,\end{aligned}\tag{2.37}$$

which is clearly bounded and (uniformly) positive on any compact subset of M_j (where we have to use that $\bar\eta\in M_j$ is not γ-degenerate). $\square$

2.5. **Conclusions.** We will draw several conclusions what we have obtained so far and what we still have to consider in the remaining part of this treatise.

2.5.1. *Cubic media in 3D.* If we consider the special case of cubic media in three space dimensions degenerate directions are given by $\bar\eta=(\bar\eta_1,\bar\eta_2,\bar\eta_3)^T$ with $\bar\eta_1^2=\bar\eta_2^2=\bar\eta_3^2$ (eight directions, corresponding to the corners of a cube) or $\bar\eta_i^2=1$ for some i (six directions, corresponding to its faces). This can be calculated directly, corresponding eigenspaces are span$\{\bar\eta\}$ and $\bar\eta^\perp=\{\xi\in\mathbb{R}^n\mid\bar\eta\cdot\xi=0\}$, or concluded by the cubic symmetry[2] of $A(\xi)$ in this particular case. See Figure 2.

To obtain the hyperbolic directions we apply Proposition 2.1 and look for the action of η under $A(\eta)$. We obtain that

(1) a direction η is hyperbolic if and only if

$$\det(\eta|A(\eta)\eta|A^2(\eta)\eta)=(\tau-\lambda-2\mu)^3\eta_1\eta_2\eta_3(\eta_1^2-\eta_2^2)(\eta_1^2-\eta_3^2)(\eta_2^2-\eta_3^2)=0,\tag{2.38}$$

thus the set of hyperbolic directions is the union of nine great circles on $\mathbb{S}^2$;

(2) $\eta||A(\eta)\eta$ for all 26 intersection points of these great circles, 14 of them are excluded as being degenerate.

Except for these 14 points on $\mathbb{S}^2$ we obtained an almost complete description of the spectrum of $B(\xi)$. We know full asymptotic expansions of eigenvalues for small and large frequencies $|\xi|$, estimates for the imaginary part of them and similar statements for eigenprojections. This information allows to draw conclusions on the large time behaviour of solutions, e.g. energy and dispersive estimates. This can be done similar to the treatment of [16], see Section 4. The remaining degenerate directions appear in two types, which can be interchanged by the action of the symmetry group. The study of these degenerate directions is what is left open so far and will be the main point of Section 3.

[2] $A(\xi)$ is invariant under the hexaeder group, i.e. the symmetry group of a cube. Thus, eigenspaces must be transferred in an appropriate way, which implies that symmetries of order 3 or 4 can only be realised by higher dimensional eigenspaces.

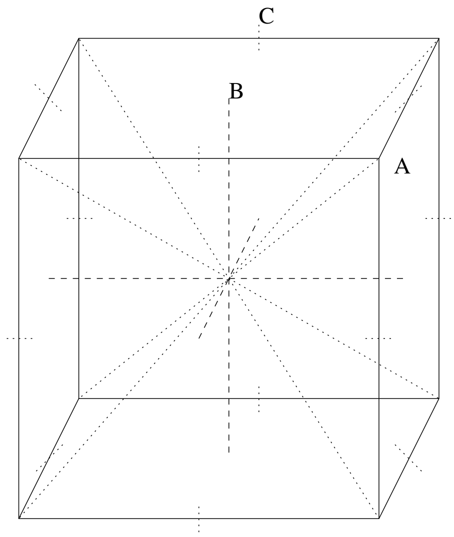

FIGURE 2. Degenerate points for cubic media correspond to symmetries of a cube. Corner points A are conic singularities, midpoints of faces B uniplanar singularities of $\operatorname{spec} A(\eta)$. The midpoints of edges C are non-degenerate, but hyperbolic with respect to two different eigenvalues.

2.5.2. *Isotropic media.* If we consider the special case of isotropic media, $A(\eta) = \mu I + (\lambda+\mu)\eta\otimes\eta$, we see that $\operatorname{spec} A(\eta) = \{\mu, \lambda+2\mu\}$ and corresponding eigenspaces are $\operatorname{span}\{\eta\}$ (corresponding to $\lambda+\mu$) and $\eta^\perp$ (corresponding to μ). All directions are (elastically) degenerate. However, we still find locally smooth systems of eigenvectors. All directions are hyperbolic and the hyperbolic eigenvalue μ has multiplicity $n-1$. Therefore the system $\mathrm{D}_t V = B(\xi)V$ decouples into a diagonal part of size $2n-2$ and a full 3×3 block and is given after a rearrangement of the entries as

$$B(t,\xi) = \begin{pmatrix} \sqrt{\mu}|\xi| &&&&&& \\ & \ddots &&&&& \\ && -\sqrt{\mu}|\xi| &&&& \\ &&& \ddots &&& \\ &&&& \sqrt{\lambda+2\mu}|\xi| && \mathrm{i}\gamma|\xi| \\ &&&&& -\sqrt{\lambda+2\mu}|\xi| & \mathrm{i}\gamma|\xi| \\ &&&& -\frac{\mathrm{i}\gamma}{2}|\xi| & -\frac{\mathrm{i}\gamma}{2}|\xi| & \mathrm{i}\kappa|\xi|^2 \end{pmatrix}. \tag{2.39}$$

This block structure corresponds to the Helmholtz decomposition of vector fields applied to the elastic displacement. If $\nabla\cdot U(t,\cdot) = 0$ the lower block cancels and we obtain wave equations with speed $\sqrt{\mu}$ for the components of U. Otherwise, if we cancel the upper block we obtain the 3×3 system corresponding to one-dimensional thermo-elasticity with its well-known properties.

2.5.3. *One-dimensional thermo-elasticity.* For completeness we mention some results on the one-dimensional system

$$u_{tt} - \tau^2 u_{xx} + \gamma\theta_x = 0, \tag{2.40a}$$

$$\theta_t - \kappa\theta_{xx} + \gamma u_{tx} = 0. \tag{2.40b}$$

We assume $\gamma, \kappa, \tau > 0$. Following our strategy we can rewrite this problem as first order system. The corresponding symbol $B(\xi)$ is given by

$$B(\xi) = \begin{pmatrix} \tau\xi & & \mathrm{i}\gamma\xi \\ & -\tau\xi & \mathrm{i}\gamma\xi \\ -\frac{\mathrm{i}}{2}\gamma\xi & -\frac{\mathrm{i}}{2}\gamma\xi & \mathrm{i}\kappa\xi^2 \end{pmatrix}. \tag{2.41}$$

Its eigenvalues satisfy asymptotic expansions for $\xi \to 0$ and $\xi \to \pm\infty$. Propositions 2.5 and 2.7 apply with $\tilde\nu^{\pm} = \pm\sqrt{\tau^2+\gamma^2}$ and

$$b_0 = \frac{\tau^2}{\tau^2+\gamma^2}, \qquad b_1 = \frac{1}{2}\frac{\gamma^2}{\tau^2+\gamma^2}. \tag{2.42}$$

Therefore, by Proposition 2.5

$$\nu_0(\xi) = \mathrm{i}\frac{\kappa\tau^2}{\tau^2+\gamma^2}\xi^2 + \mathcal{O}(\xi^3), \tag{2.43a}$$

$$\nu_1^{\pm}(\xi) = \pm\sqrt{\tau^2+\gamma^2}\xi + \mathrm{i}\frac{\kappa\gamma^2}{2(\tau^2+\gamma^2)}\xi^2 + \mathcal{O}(\xi^3), \tag{2.43b}$$

as $\xi \to 0$ and by Proposition 2.7

$$\nu_0(\xi) = \mathrm{i}\kappa\xi^2 - \mathrm{i}\frac{\gamma}{\kappa} + \mathcal{O}(\xi^{-1}), \tag{2.43c}$$

$$\nu_1^{\pm}(\xi) = \pm\tau\xi + \mathrm{i}\frac{\gamma^2}{2\kappa} + \mathcal{O}(\xi^{-1}), \tag{2.43d}$$

as $\xi \to \infty$. The essential information for large time estimates is given by the behaviour of the imaginary part. It follows that $\operatorname{Im}\nu(\xi) > C_\epsilon$ for $|\xi| \geq \epsilon$ for certain constants and

$$\operatorname{Im}\nu_0(\xi) \sim \frac{\kappa\tau^2}{\tau^2+\gamma^2}\xi^2, \qquad \operatorname{Im}\nu_1^{\pm}(\xi) \sim \frac{\kappa\gamma^2}{2(\tau^2+\gamma^2)}\xi^2, \qquad \xi \to 0. \tag{2.44}$$

2.5.4. *Hexagonal media in 3D.* For hexagonal media in three space dimensions the situation is (surprisingly) simpler than for cubic media. The elastic operator defined by (1.4)–(1.5) is invariant under rotation around the x_3-axis (taking into account a corresponding rotation of the reference frame for vectors) and therefore it suffices to understand its cross sections in the x_1–x_2 plane. We will sketch some of the properties of the corresponding symbol $A(\eta)$.

Following Proposition 2.1 we obtain

(1) that

$$\det(\eta|A(\eta)\eta|A^2(\eta)\eta) = 0, \tag{2.45}$$

such that *all* directions $\eta \in \mathbb{S}^2$ are hyperbolic. The corresponding eigenspace is (generically) given by multiples of $(\eta_2, -\eta_1, 0)$ such that the hyperbolic eigenvalue is

$$\frac{\tau_1 - \lambda_1}{2}(\eta_1^2 + \eta_2^2) + \mu\eta_3^2. \tag{2.46}$$

(2) It remains to look for directions with two hyperbolic eigenvalues. They satisfy $\eta || A(\eta)$. This is true, if $\eta_3 = 0$ or if $\eta_1 = \eta_2 = 0$ or if

$$\eta_3^2 = \frac{\lambda_2 + 2\mu - \tau_1}{2\lambda_2 + 4\mu + \tau_1 - \tau_2}, \tag{2.47}$$

provided the latter expression is non-negative. Except in the limiting case $\tau_1 = \lambda_2 + 2\mu$, the coupling functions vanish to first order along the corresponding circle. If $\tau_1 = \tau_2 = \lambda_2 + 2\mu$ all directions are hyperbolic with two hyperbolic eigenvalues and if $\tau_1 = \lambda_2 + 2\mu \neq \tau_2$ coupling functions vanish to third order.

(3) The matrices $A(\eta)$ are invariant under rotation. Introducing spherical coordinates on $\mathbb{S}^2$

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos\phi \cos\psi + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin\phi \cos\psi + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin\psi \tag{2.48}$$

and using a corresponding (moving) basis for vectors given by

$$\frac{\pm 1}{\sqrt{\eta_1^2 + \eta_2^2}} \begin{pmatrix} \eta_2 \\ -\eta_1 \\ 0 \end{pmatrix}, \qquad \eta, \qquad \frac{\pm \eta_3}{\sqrt{1 - \eta_3^2}} \begin{pmatrix} \eta_1 \\ \eta_2 \\ -\frac{\eta_1^2 + \eta_2^2}{\eta_3} \end{pmatrix} \tag{2.49}$$

(sign chosen to make them smoothly dependent on $\eta \neq \pm(0,0,1)^\top$) decomposes $A(\eta)$ into (1,2)-block-diagonal structure (independent of the co-ordinate ϕ). The scalar block corresponds to the eigenvalue (2.46), while the 2×2 block has trace $\mu + \tau_1 \cos^2\psi + \tau_2 \sin^2\psi$ and determinant $\mu\tau_1 \cos^4\psi + \mu\tau_2 \sin^4\psi + \frac{\tau_1\tau_2 - 2\lambda_2 - \lambda_2^2}{4} \sin^2 2\psi$.

If $(\tau_1 - \mu)(\tau_2 - \mu) \neq 0$, the 2×2 block has distinct eigenvalues for all ψ and therefore the only degenerate directions are directions where this block has (2.46) as one of its eigenvalues. This happens if and only if the right hand side of (2.47) is non-negative and on the circle defined by that equation.

Thus, the previously developed theory is applicable for all directions except the degenerate ones $\eta_1 = \eta_2 = 0$ or (2.47). The always existent hyperbolic eigenvalue (2.46) leads to a decoupling of the thermo-elastic system into two scalar blocks and a (at least formally) 2D thermo-elastic system.

Due to rotational invariance, it suffices to treat the cut $\eta_1 = 0$ for handling of degenerate directions. This will be sketched later.

3. Some special degenerate directions

We want to study neighbourhoods of degenerate directions for some particular cases. To study degenerate directions in full generality is beyond the scope of this paper. We relate our approach to the type of singularity of the corresponding *Fresnel surface*

$$\mathcal{S} = \{\xi \in \mathbb{R}^n \,|\, 1 \in \operatorname{spec} A(\xi)\}. \tag{3.1}$$

This surface is in general n-sheeted and for all non-degenerate directions these sheets are given by

$$\mathcal{S}_j = \{\xi \in \mathbb{R}^n \text{ non-deg.} \,|\, \omega_j(\xi) = 1\} = \{\omega_j^{-1}(\eta)\eta \,|\, \eta \in \mathbb{S}^{n-1} \text{ non-degenerate}\}, \tag{3.2}$$

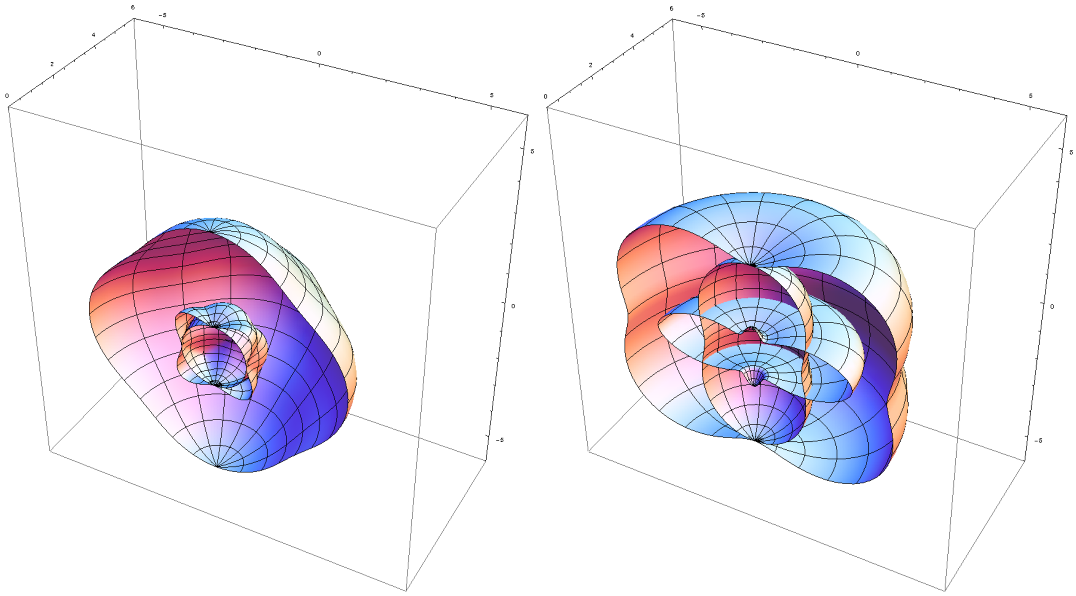

FIGURE 3. A cut through the Fresnel surfaces for examples of a cubic and a hexagonal medium. The material parameter are $\lambda = 1$, $\tau = 4$ and $\mu = 1$ for the picture on the left (cubic) and $\lambda_1 = 1$, $\lambda_2 = \frac{1}{5}$, $\tau_1 = 4$, $\tau_2 = 1$ and $\mu = 3$ for the picture on the right (hexagonal).

while in degenerate points the surface is self-intersecting. For the importance of these surfaces in elasticity theory and some interesting properties of them we refer to Duff [4] or the investigations from Musgrave [12], [13] and Miller-Musgrave [14].

We remark only one of the general properties of $\mathcal{S}$ here. If $A(\xi)$ is polynomial in ξ then the surface $\mathcal{S}$ is algebraic of degree $2n$ and therefore any straight line intersecting $\mathcal{S}$ has at most $2n$ intersection points with $\mathcal{S}$. In particular, if the inner sheet $\mathcal{S}_n$ does not touch any of the the outer sheets, it has to be strictly convex.

3.1. **General strategy.** If we investigate isolated degenerate directions or regular manifolds of degenerate directions of codimension greater than one we are faced with two major obstacles. Generically, eigenvectors of $A(\eta)$ can not be chosen continuously in a neighbourhood of the degenerate direction and therefore a reformulation as system of first order as in (2.7) is problematic. This problem is related to higher-dimensional perturbation theory of matrices. It is well-known that in the one-dimensional situation eigenspaces are continuous (see, e.g., the book of Kato, [6]) and it can be resolved by introducing polar co-ordinates / normal co-ordinates around the degenerate directions and a system related to (2.7) can be formulated on a corresponding blown-up space (see, e.g., (3.10) below). Second obstacle are the multiplicities itself. Eigenvalues and eigenvectors of the constructed system of first order do not possess asymptotic expansions in powers of $|\xi|$ as $|\xi|$ tends to 0 or ∞. However, especially in the three-dimensional setting we can write full asymptotic expansions in the distance to the degeneracy uniform in all remaining co-ordinates.

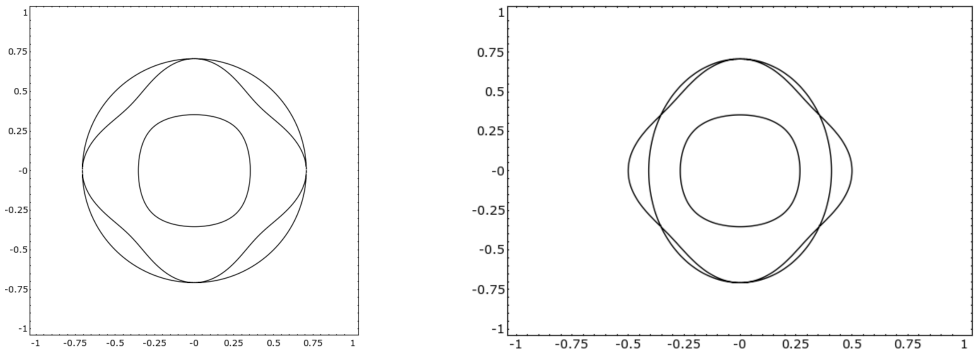

FIGURE 4. Cuts of the Fresnel surface for cubic media; on the left hand side in the plane $\eta_3 = 0$, on the right for $\eta_1 = \eta_2$. The parameters are chosen as $\tau = 8$, $\lambda = 2$ and $\mu = 2$.

We will discuss the application of this general strategy in detail for conic singularities of the Fresnel surface appearing for the case of cubic media and give the corresponding results for uniplanar singularities afterwards. Finally we will consider hexagonal media and show that they are much simpler in their analytical structure.

3.2. **Cubic media, conic singularities.** The Fresnel surface for cubic media has eight conic singularities which are related by the symmetries of the underlying medium. It suffices to consider one of them and we choose $\bar\eta = \frac{1}{\sqrt3}(1,1,1)^T \in \mathbb{S}^2$. Near this direction we introduce polar co-ordinates (ϵ,ϕ) on the sphere $\mathbb{S}^2$ by

$$\eta = \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix} = \sqrt{1-\epsilon^2}\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \epsilon\frac{1}{\sqrt6}\begin{pmatrix}-1\\-1\\2\end{pmatrix}\cos\phi + \epsilon\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\sin\phi. \tag{3.3}$$

They allow to blow up the singularity by looking at $[0,\infty)\times\mathbb{S}^1$ instead of $\mathbb{R}^2$ as local model of $\mathbb{S}^2$. In order to simplify notation, we apply a diagonaliser $\tilde M$ of $A(\bar\eta)$ to our matrices. For this we choose the unitary matrix

$$\tilde M = \frac{1}{\sqrt6}\begin{pmatrix}\sqrt2 & -1 & \sqrt3\\ \sqrt2 & -1 & -\sqrt3\\ \sqrt2 & 2 & 0\end{pmatrix} \tag{3.4}$$

(corresponding to the vectors chosen already in (3.3)). The matrix $\tilde M^{-1}A(\eta)\tilde M$ has a full asymptotic expansion as $\epsilon\to0$ and can be written as

$$\tilde M^{-1}A(\epsilon,\phi)\tilde M = A_0 + \epsilon A_1(\phi) + \mathcal{O}(\epsilon^2), \qquad \epsilon\to0 \tag{3.5}$$

with matrices

$$A_0 = \operatorname{diag}\left(\frac{\tau+2\lambda+4\mu}{3}, \frac{\tau+\mu-\lambda}{3}, \frac{\tau+\mu-\lambda}{3}\right), \tag{3.6a}$$

$$\begin{aligned} A_1(\phi) = \frac{2\tau-\mu+\lambda}{3} &\begin{pmatrix} & \cos\phi & \sin\phi \\ \cos\phi & & \\ \sin\phi & & \end{pmatrix} \\ &+ \frac{\sqrt{2}(-\tau+2\mu+\lambda)}{3}\begin{pmatrix} 0 & & \\ & -\cos\phi & \sin\phi \\ & \sin\phi & \cos\phi \end{pmatrix}. \end{aligned} \tag{3.6b}$$

Now we can apply the block-diagonalisation procedure (again following [5, Sec. 2.2]) to obtain the behaviour of eigenvalues and eigenprojections of $\tilde{M}^{-1}A(\epsilon,\phi)\tilde{M}$ as $\epsilon \to 0$ for all ϕ. We restrict consideration to the case where $\lambda + \mu \neq 0$, such that A_0 has two different eigenvalues.

Proposition 3.1. *The eigenvalues $\varkappa_j(\epsilon,\phi)$ and the corresponding eigenprojections of $A(\epsilon,\phi)$ have uniformly in ϕ full asymptotic expansions as $\epsilon \to 0$. The main terms are given by*

$$\varkappa_1(\epsilon,\phi) = \frac{\tau+2\lambda+4\mu}{3} + \mathcal{O}(\epsilon^2), \tag{3.7a}$$

$$\varkappa_2(\epsilon,\phi) = \frac{\tau+\mu-\lambda}{3} + \frac{\sqrt{2}(-\tau+2\mu+\lambda)}{3}\epsilon + \mathcal{O}(\epsilon^2), \tag{3.7b}$$

$$\varkappa_3(\epsilon,\phi) = \frac{\tau+\mu-\lambda}{3} - \frac{\sqrt{2}(-\tau+2\mu+\lambda)}{3}\epsilon + \mathcal{O}(\epsilon^2). \tag{3.7c}$$

Remark 3.1. The exceptional case when $\tau = \lambda + 2\mu$ corresponds to isotropic media and is therefore excluded. In all other cases the two sheets $\omega_2(\eta) = \sqrt{\varkappa_2(\eta)}$ and $\omega_3(\eta) = \sqrt{\varkappa_3(\eta)}$ form a double cone on the Fresnel surface S. Hence, the statement explains the notion of conical singularity. Note, that the linear terms are independent of ϕ and therefore the cone is approximately a spherical cone near the conic point.

Proof. We will only shortly review the main steps. First we (1,2)-block-diagonalise $\tilde{M}^{-1}A(\epsilon,\phi)\tilde{M}$ (modulo $\mathcal{O}(\epsilon^N)$ for any N we like). The diagonaliser we are going to construct has the form $I + \epsilon N_1^{(1)}(\phi) + \dots + \epsilon^{N-1}N_1^{(N-1)}(\phi)$ and as in [5, Sec. 2.2] its terms are given by recursion formulae. For $N_1^{(1)}$ we divide the off-(block-)diagonal terms of A_1 by the difference of the corresponding diagonal entries of A_0. This gives as first term

$$N_1^{(1)}(\phi) = \frac{2\tau-\mu+\lambda}{3(\lambda+\mu)}\begin{pmatrix} & \cos\phi & \sin\phi \\ -\cos\phi & & \\ -\sin\phi & & \end{pmatrix} \tag{3.8}$$

and allows to cancel the off-(block-)diagonal entries of A_1. We skip the further construction and move to the next step. Since the lower 2×2 block of $\operatorname{b-diag}_{1,2} A_1$ has distinct eigenvalues (namely ± 1) we can now perform a diagonalisation scheme in the subspace corresponding to this block (modulo $\mathcal{O}(\epsilon^N)$). Again we restrict ourselves to the main

terms. A unitary diagonaliser of the 2×2-block can be chosen as the unitary matrix

$$\tilde{M}_2(\phi) = \begin{pmatrix} 1 & & \\ & \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \\ & \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \end{pmatrix}. \tag{3.9}$$

After transforming with that matrix we apply the recursive scheme to diagonalise further. Note that after applying $\tilde{M}_2(\phi)$ the matrix is diagonal modulo $\mathcal{O}(\epsilon^2)$ and therefore, $\tilde{M}(I+\epsilon N_1^{(1)}(\phi))\tilde{M}_2(\phi) = M_0(\phi)+\epsilon M_1(\phi)+\mathcal{O}(\epsilon^2)$ determines the main terms of a diagonaliser of the matrix $A(\epsilon,\phi)$ and we can deduce the statements about the eigenvalue asymptotics. □

3.2.1. *System formulation.* Let $M(\epsilon,\phi)$ be the diagonaliser of $A(\epsilon,\phi)$ constructed in Proposition 3.1. Then we consider

$$V(t,\epsilon,\phi,|\xi|) = \begin{pmatrix} (\mathrm{D}_t + |\xi|\mathcal{D}^{1/2}(\epsilon,\phi))M^{-1}(\epsilon,\phi)\hat{U}(t,\xi) \\ (\mathrm{D}_t - |\xi|\mathcal{D}^{1/2}(\epsilon,\phi))M^{-1}(\epsilon,\phi)\hat{U}(t,\xi) \\ \hat{\theta} \end{pmatrix} \in \mathbb{C}^7, \tag{3.10}$$

with $\xi = |\xi|\eta(\epsilon,\phi)$ and $\mathcal{D}^{1/2}(\epsilon,\phi) = \operatorname{diag}(\omega_1(\epsilon,\phi),...)$ the diagonal matrix containing the square roots $\omega_j(\epsilon,\phi) = \sqrt{\varkappa_j(\epsilon,\phi)}$ of the eigenvalues of $A(\epsilon,\phi)$. The vector V satisfies the first order system $\mathrm{D}_tV = B(\epsilon,\phi,|\xi|)V$ with $B(\epsilon,\phi,|\xi|) = B_1(\epsilon,\phi)|\xi| + B_2|\xi|^2$ given by

$$B_1(\epsilon,\phi) = \begin{pmatrix} \omega_1(\epsilon,\phi) & & & & \mathrm{i}\gamma a_1(\epsilon,\phi) \\ & \omega_2(\epsilon,\phi) & & & \mathrm{i}\gamma a_2(\epsilon,\phi) \\ & & \ddots & & \vdots \\ & & & -\omega_3(\epsilon,\phi) & \mathrm{i}\gamma a_3(\epsilon,\phi) \\ -\frac{\mathrm{i}}{2}\gamma a_1(\epsilon,\phi) & -\frac{\mathrm{i}}{2}\gamma a_2(\epsilon,\phi) & \cdots & -\frac{\mathrm{i}}{2}\gamma a_3(\epsilon,\phi) & 0 \end{pmatrix} \tag{3.11}$$

and $B_2 = \operatorname{diag}(0,\dots,0,\mathrm{i}\kappa)$. The coupling functions $a_j(\epsilon,\phi)$ are the components of the vector $M^{-1}(\epsilon,\phi)\eta$. From Proposition 3.1 we know that they have asymptotic expansions as $\epsilon\to 0$.

Remark 3.2. *1.* Since $M^{-1}(\epsilon,\phi) = \tilde{M}_2^*(\phi)(I-\epsilon N_1^{(1)}(\phi))\tilde{M}^* + \mathcal{O}(\epsilon^2)$ by our construction it follows that

$$a_1(\epsilon,\phi) = 1+\mathcal{O}(\epsilon^2), \tag{3.12a}$$

$$a_2(\epsilon,\phi) = \epsilon\frac{2(\mu+2\lambda-\tau)}{3(\lambda+\mu)}\sin\frac{3\phi}{2} + \mathcal{O}(\epsilon^2), \tag{3.12b}$$

$$a_3(\epsilon,\phi) = \epsilon\frac{2(\mu+2\lambda-\tau)}{3(\lambda+\mu)}\cos\frac{3\phi}{2} + \mathcal{O}(\epsilon^2). \tag{3.12c}$$

We know that the coupling functions vanish along three great circles through $\bar{\eta}$. We see that the numbering of the eigenprojections is not consistent along the circles. The coupling functions a_2 and a_3 vanish both in the degenerate direction.
2. Since we do not assume that $M(\epsilon,\phi)$ is unitary the relation $\sum_j a_j^2 = 1$ does not hold for these coupling functions. However, $M_0(\phi)$ is unitary and therefore $\sum_j a_j^2 = 1+\mathcal{O}(\epsilon)$ as already observed.

3.2.2. *Real and imaginary parts of eigenvalues.* The coefficient matrix $B(\epsilon,\phi,|\xi|)$ has the same structure as $B(\xi)$ in Section 2. Therefore, we can conclude similar statements on eigenvalues and their behaviour by (a) investigating the characteristic polynomial and (b) block-diagonalising for small and large $|\xi|$, respectively.

Proposition 3.2. (1) $\operatorname{tr} B(\epsilon,\phi,|\xi|) = \mathrm{i}\kappa|\xi|^2$ *and* $\det B(\epsilon,\phi,|\xi|) = \mathrm{i}\kappa|\xi|^2 \det A(\xi)$.

(2) $B(\epsilon,\phi,|\xi|)$ *has purely real eigenvalues for* $|\xi|\neq 0$ *if and only if* $a_2(\epsilon,\phi)a_3(\epsilon,\phi) = 0$, *i.e.,* $\epsilon = 0$ *or* $\phi\in\frac{\pi}{3}\mathbb{Z}$.

(3) $B(0,\phi,|\xi|)$ *has the real eigenvalues* $\pm\omega_{2,3}(0,\phi) = \frac{\sqrt{3}}{3}(\tau+\mu-\lambda)$ *and three eigenvalues satisfying* $\operatorname{Im}\nu \geq C$ *if* $|\xi|\geq c$ *and* $\operatorname{Im}\nu \sim |\xi|^2$ *if* $|\xi|<c$.

(4) *The quotient*

$$\frac{a_2^2(\epsilon,\phi)(\nu_{2,3}^2(\epsilon,\phi,|\xi|)-\varkappa_3(\epsilon,\phi)|\xi|^2)+a_3^2(\epsilon,\phi)(\nu_{2,3}^2(\epsilon,\phi,|\xi|)-\varkappa_2(\epsilon,\phi)|\xi|^2)}{(\nu_{2,3}^2(\epsilon,\phi,|\xi|)-\varkappa_2(\epsilon,\phi)|\xi|^2)(\nu_{2,3}^2(\epsilon,\phi)-\varkappa_3(\epsilon,\phi)|\xi|^2)} \tag{3.13}$$

involving the hyperbolic eigenvalues $\nu_{2,3}^{\pm}$ *of* $B(\epsilon,\phi,|\xi|)$ *is smooth and non-vanishing for fixed values of* $|\xi|$.

Proof. We consider only part (2) to (4). The characteristic polynomial of B is given by an expression like (2.18). If we assume that eigenvalues are purely real we can split the expression into real and imaginary part. We consider the imaginary part first, which leads to

$$\kappa|\xi|^2\prod_{j=1}^{3}(\nu^2-\varkappa_j(\epsilon,\phi)|\xi|^2)=0. \tag{3.14}$$

Therefore, real eigenvalues have to coincide with the square roots of eigenvalues of $A(\xi)$. If we assume $\nu^2 = \varkappa_j(\epsilon,\phi)|\xi|^2)$ is a root of the characteristic equation, we can divide by the corresponding factor and obtain if $\epsilon\neq 0$ (and therefore A is non-degenerate)

$$a_j^2(\epsilon,\phi)=0. \tag{3.15}$$

If $\epsilon=0$ the characteristic polynomial factors as

$$((\nu-\mathrm{i}\kappa|\xi|^2)(\nu^2-\bar\varkappa_1|\xi|^2)-\nu\gamma^2|\xi|^2)(\nu-\bar\varkappa_{2,3}|\xi|^2)^2=0 \tag{3.16}$$

with $\bar\varkappa_1 = \frac{1}{3}(\tau+2\lambda+4\mu)$ and $\bar\varkappa_{2,3}=\frac{1}{3}(\tau+\mu-\lambda)$. The first factor resembles one-dimensional thermo-elasticity (with $\tau^2=\bar\varkappa_1$) and gives three roots with positive imaginary parts subject to (2.43) and (2.44). Finally (4) follows by collecting the two related terms in the characteristic equation of form (2.19). The imaginary part of the quotient is given by $-\kappa|\xi|^2/\nu_{2,3}^{\pm}$ in hyperbolic/degenerate directions and therefore non-zero. □

The quotient (3.13) may be used to determine asymptotic expansions of the hyperbolic eigenvalue and its imaginary part as $\epsilon\to 0$ for fixed $|\xi|$ and $\phi\notin\frac{\pi}{3}\mathbb{Z}$. We will follow a different strategy and diagonalise as $\epsilon\to 0$ uniform on bounded ξ.

3.2.3. *Asymptotic expansion as* $\epsilon\to 0$ *uniform in* $|\xi|$. Note first, that $B(|\xi|,0,\phi)$ is independent of ϕ and just the system of one-dimensional thermo-elasticity (2.41) extended by four additional diagonal entries. Since we need to understand this system first, we are going to recall some facts about the one-dimensional theory. As $|\xi|$ becomes small/large we already gave asymptotic expansions of eigenvalues in Section 2.5.3. The bit of information which is still missing is contained in the following lemma.

Lemma 3.3. *The coefficient matrix $B(\xi)$ of the one-dimensional thermo-elastic system given in* (2.41) *has for $\xi \neq 0$ and under the natural assumptions $\gamma, \kappa, \tau > 0$ only simple eigenvalues.*

Proof. Note that the characteristic polynomial of this matrix $B(\xi)$ is given by

$$\nu^3 - \mathrm{i}\kappa|\xi|^2\nu^2 + \tau^2|\xi|^2\nu + \mathrm{i}\tau^2\kappa|\xi|^4,$$

which is invariant under the transform $\nu \mapsto \overline{-\nu}$ and has alternating imaginary and real coefficients. From that we conclude that the only possible solutions are of the form $\mathrm{i}a$, $b+\mathrm{i}c$ and $-b+\mathrm{i}c$ for certain real a, b, c. Furthermore, from the general theory of Section 2 it is clear that $a, c > 0$. Thus, the only possibility for multiplicities to occur is if $b = 0$. Plugging in $b = 0$ and multiplying the corresponding linear factors gives

$$\nu^3 - \nu^2(\mathrm{i}a + 2\mathrm{i}c) - \nu(c^2 + 2ac) + \mathrm{i}c^2a.$$

Comparing coefficients with the above polynomial implies that $\kappa|\xi|^2 = -ca/(c + 2a)$, which contradicts to the positivity of all quantities involved. $\square$

We write the coefficient matrix $B(|\xi|, \epsilon, \phi)$ as sum of homogeneous components in ϵ

$$|\xi|^{-1}B(|\xi|, \epsilon, \phi) = B^{(0)}(|\xi|, \phi) + \epsilon B^{(1)}(|\xi|, \phi) + \mathcal{O}(\epsilon^2), \tag{3.17}$$

where

$$B^{(0)}(|\xi|, \phi) = \begin{pmatrix} \bar\omega_1 & & & & & & \mathrm{i}\gamma \\ & \bar\omega_2 & & & & & \\ & & \bar\omega_2 & & & & \\ & & & -\bar\omega_1 & & & \mathrm{i}\gamma \\ & & & & -\bar\omega_2 & & \\ & & & & & -\bar\omega_2 & \\ -\frac{\mathrm{i}}{2}\gamma & & & -\frac{\mathrm{i}}{2}\gamma & & & \mathrm{i}\kappa|\xi| \end{pmatrix}, \tag{3.18}$$

$$B^{(1)}(|\xi|, \phi) = \begin{pmatrix} 0 & & & & & & 0 \\ & \delta_1 & & & & & \mathrm{i}\gamma\delta_2 \sin\frac{3\phi}{2} \\ & & -\delta_1 & & & & \mathrm{i}\gamma\delta_2 \cos\frac{3\phi}{2} \\ & & & 0 & & & 0 \\ & & & & \delta_1 & & \mathrm{i}\gamma\delta_2 \sin\frac{3\phi}{2} \\ & & & & & -\delta_1 & \mathrm{i}\gamma\delta_2 \cos\frac{3\phi}{2} \\ 0 & -\frac{\mathrm{i}\gamma\delta_2}{2}\sin\frac{3\phi}{2} & -\frac{\mathrm{i}\gamma\delta_2}{\cos}\frac{3\phi}{2} & 0 & -\frac{\mathrm{i}\gamma\delta_2}{2}\sin\frac{3\phi}{2} & -\frac{\mathrm{i}\gamma\delta_2}{2}\cos\frac{3\phi}{2} & 0 \end{pmatrix}, \tag{3.19}$$

and $\bar\omega_1 = \sqrt{\frac{\tau+2\lambda+4\mu}{3}}$, $\bar\omega_2 = \sqrt{\frac{\tau+\mu-\lambda}{3}}$, $\delta_1 = \frac{1}{\sqrt{6}}\frac{-\tau+2\mu+\lambda}{\sqrt{\tau+\mu-\lambda}}$ and $\delta_2 = \frac{2(\mu+2\lambda-\tau)}{3(\lambda+\mu)}$.

As a direct consequence of the previous lemma in combination with the asymptotic expansions of Section 2.5.3 we obtain

Proposition 3.4. *Assume, that $\lambda + \mu \neq 0$ and $\gamma^2 + \lambda + \mu \neq 0$. Then the matrix $B^{(0)}(|\xi|, \phi)$ has uniformly separated eigenvalues in $|\xi| \in \mathbb{R}$, $\phi \in \mathbb{S}^1$ (where $\pm\bar\omega_2$ are of constant multiplicity two).*

Now we can apply several steps of diagonalisation based on the scheme of [5, Sec. 2]. At first we apply the diagonaliser of the main part. This has only effects on the two

entries related to $\bar{\omega}_1$ and the last row/column and determines the eigenvalues $\nu_0(|\xi|,\epsilon,\phi)$ and $\nu_1^\pm(|\xi|,\epsilon,\phi)$ modulo ϵ^2. Furthermore, Proposition 3.4 allows to $(1,2,1,2,1)$-block-diagonalise modulo $\mathcal{O}(\epsilon^N)$, N arbitrary.

Finally we can investigate the remaining 2×2-blocks and diagonalise again because the ϵ-homogeneous entries $\pm\delta_1\epsilon$ are distinct (trivially uniform in $|\xi|$ and ϕ).

Proposition 3.5. *Assume, that $\lambda+\mu\neq 0$ and $\gamma^2+\lambda+\mu\neq 0$. The eigenvalues of $B(|\xi|,\epsilon,\phi)$ have uniformly in $|\xi|$ and ϕ full asymptotic expansions as $\epsilon\to 0$. The first main terms are given as*

$$\nu_0(|\xi|,\epsilon,\phi) = \check{\nu}_0(|\xi|) + |\xi|\mathcal{O}(\epsilon^2), \tag{3.20a}$$

$$\nu_1^\pm(|\xi|,\epsilon,\phi) = \check{\nu}_1^\pm(|\xi|) + |\xi|\mathcal{O}(\epsilon^2), \tag{3.20b}$$

$$\nu_{2/3}^{\pm_1,\pm_2}(|\xi|,\epsilon,\phi) = \pm_1\bar{\omega}_2|\xi| \pm_2 \delta_1|\xi|\epsilon + |\xi|\mathcal{O}(\epsilon^2) \tag{3.20c}$$

where $\check{\nu}_0(|\xi|)$ and $\check{\nu}_1^\pm(|\xi|)$ are the eigenvalues of the one-dimensional thermo-elastic system with propagation speed $\bar{\omega}_1$ and the signs $\pm_1$ and $\pm_2$ are independent of each other.

Remark 3.3. The statement holds true uniform in $|\xi|$. However, it is only of use as long as the error terms $|\xi|\epsilon^N$ are smaller than the size of the eigenvalues. For $|\xi|\to 0$ the eigenvalues of the one-dimensional thermo-elastic system behave like $\check{\nu}_0(|\xi|)\sim|\xi|^2$ and $\check{\nu}_1^\pm(|\xi|)\sim\pm|\xi|$. Hence, the statement of (3.20a) is only of use if $|\xi|\epsilon^2\ll|\xi|^2$, i.e. if $\epsilon^2\ll|\xi|$. For $|\xi|\to\infty$ we know similarly $\check{\nu}_0(|\xi|)\sim|\xi|^2$ and $\check{\nu}_1^\pm(|\xi|)\sim\pm|\xi|$, which in turn implies that the expansion determines the behaviour of the eigenvalues.

This restriction is by no means a severe one; the expansion is only of interest for the 'degenerate' eigenvalues $\nu_{2/3}^{\pm_1,\pm_2}(|\xi|,\epsilon,\phi)$ (for which no such restriction appears).

3.2.4. *Diagonalisation for small and large $|\xi|$.* To complete the picture we want to give some comments on expansions for small and large values of $|\xi|$ under the same assumptions as in Proposition 3.5. Using the ideas from [23] we can employ the (block) diagonalisation scheme to separate the three non-degenerate eigenvalues from the two degenerate ones asymptotically and give full asymptotic expansions for them as $|\xi|$ tends to zero or infinity. The obtained expressions coincide with the formulae from Propositions 2.5 and 2.7. It remains to understand the behaviour of the remaining 2×2-blocks. This can be done directly by solving the characteristic polynomial as in [16, Prop. 2.7] or by a second diagonalisation scheme.

We focus on the latter idea and consider the case of small $|\xi|$ first. The 2×2-blocks have the form

$$f_0(|\xi|,\epsilon,\phi)I + \begin{pmatrix}\delta_0(|\xi|,\epsilon,\phi) & \\ & -\delta_0(|\xi|,\epsilon,\phi)\end{pmatrix} + \mathcal{O}(|\xi|^2) \tag{3.21}$$

with a function $\delta_0(|\xi|,\epsilon,\phi)\sim\epsilon|\xi|$. If we restrict the consideration to the zone

$$\mathcal{Z}_0(c) = \{(|\xi|,\epsilon,\phi)\,:\,|\xi|\le c\epsilon,\ \epsilon\ll 1\}, \tag{3.22}$$

the remainder can be written as $\epsilon|\xi|\mathcal{O}(\epsilon^{-1}|\xi|)$ and the standard diagonalisation scheme applied to the last two terms gives full asymptotic expansions in powers of $\epsilon^{-1}|\xi|$ as $\epsilon^{-1}|\xi|\to 0$,

$$f_0(|\xi|,\epsilon,\phi)\pm\delta_0(|\xi|,\epsilon,\phi)+\ldots.+\epsilon|\xi|\mathcal{O}(\epsilon^{-N}|\xi|^N). \tag{3.23}$$

A similar idea applies for large $|\xi|$ in the zone

$$\mathcal{Z}_\infty(N) = \{(|\xi|, \epsilon, \phi) \,:\, \epsilon|\xi| \geq N,\, \epsilon \ll 1\}. \tag{3.24}$$

Based on

$$f_\infty(|\xi|, \epsilon, \phi) I + \begin{pmatrix} \delta_\infty(|\xi|, \epsilon, \phi) & \\ & -\delta_\infty(|\xi|, \epsilon, \phi) \end{pmatrix} + \mathcal{O}(1) \tag{3.25}$$

with a function $\delta_\infty(|\xi|, \epsilon, \phi) \sim \epsilon|\xi|$ it gives asymptotic expansions in powers of $\epsilon|\xi|$ as $\epsilon|\xi| \to \infty$.

3.3. **Cubic media, uniplanar singularities.** The Fresnel surface for cubic media has six uniplanar singularities. Again they are equivalent and it suffices to consider the neighbourhood of $\bar\eta = (1, 0, 0)^T \in \mathbb{S}^2$.

We introduce polar co-ordinates near $\bar\eta$. Let $\epsilon \geq 0$ and $\phi \in [-\pi, \pi)$. Then we set

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \sqrt{1-\epsilon^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \epsilon \cos\phi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \sin\phi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.26}$$

and use an asymptotic expansion of $A(\eta)$ as $\epsilon \to 0$

$$A(\eta) = A_0 + \epsilon A_1(\phi) + \epsilon^2 A_2(\phi) + \mathcal{O}(\epsilon^3) \tag{3.27}$$

with coefficients

$$A_0 = \operatorname{diag}(\tau, \mu, \mu) \tag{3.28a}$$

$$A_1(\phi) = (\lambda + \mu) \begin{pmatrix} & \cos\phi & \sin\phi \\ \cos\phi & & \\ \sin\phi & & \end{pmatrix} \tag{3.28b}$$

$$A_2(\phi) = (\tau - \mu) \begin{pmatrix} -1 & & \\ & \cos^2\phi & \\ & & \sin^2\phi \end{pmatrix} + \frac{\lambda+\mu}{2} \begin{pmatrix} 0 & & \\ & & \sin 2\phi \\ & \sin 2\phi & \end{pmatrix} \tag{3.28c}$$

to deduce properties of the eigenvalues and eigenprojections of $A(\eta)$ near $\bar\eta$. We restrict considerations to the case when $\tau \neq \mu$. Then the following statement follows again by the two-step diagonalisation procedure (like in the conical case and as developed in [5], [16]).

Proposition 3.6. *Assume $\lambda+\mu \neq 0$, $\tau \neq \mu$ and $\tau \neq \lambda+2\mu$. Then the eigenvalues $\varkappa_j(\eta)$ and the corresponding eigenprojections have uniformly in ϕ full asymptotic expansions as $\epsilon \to 0$. The main terms are given by*

$$\varkappa_1(\eta) = \tau - C\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{3.29a}$$

$$\varkappa_2(\eta) = \mu + \frac{1}{2}\left(C + \sqrt{C^2 - (C^2 - D^2)\sin^2(2\phi)}\right)\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{3.29b}$$

$$\varkappa_3(\eta) = \mu + \frac{1}{2}\left(C - \sqrt{C^2 - (C^2 - D^2)\sin^2(2\phi)}\right)\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{3.29c}$$

where

$$C = \frac{(\tau-\mu)^2 - (\lambda+\mu)^2}{\tau - \mu}, \qquad D = \lambda + \mu \tag{3.30}$$

Remark 3.4. This statement reflects what we mean by an uniplanar singularity. Two of the eigenvalues coincide up to second order.

Proof. We follow the diagonalisation scheme. A_0 is already diagonal, A_1 does not contain (1,2)-block diagonal entries. To get expansions for the eigenvalues we have to apply two steps of block-diagonalisation. First we treat A_1 by the aid of

$$N_1^{(1)}(\phi) = \frac{\lambda+\mu}{\tau-\mu} \begin{pmatrix} & -\cos\phi & -\sin\phi \\ \cos\phi & & \\ \sin\phi & & \end{pmatrix}, \tag{3.31}$$

such that $I + \epsilon N_1^{(1)}(\phi)$ block-diagonalises the matrix modulo ϵ^2. This preserves A_0 and $0 = \text{b-diag}_{1,2} A_1$ and gives the new 2-homogeneous component

$$A_2 + A_1 N_1^{(1)}, \qquad A_1 N_1^{(1)} = \frac{(\lambda+\mu)^2}{\tau-\mu} \operatorname{diag}(1, -\cos^2\phi, -\sin^2\phi). \tag{3.32}$$

The starting terms of the expansion of the first eigenvalue can be read directly from these matrices. For the remaining two we have to diagonalise the lower 2×2 block. This block has the form

$$\begin{pmatrix} C\cos^2\phi & D\sin\phi\cos\phi \\ D\sin\phi\cos\phi & C\sin^2\phi \end{pmatrix} \tag{3.33}$$

with C, D from (3.30). Eigenvalues of this matrix are uniformly separated if the condition

$$C^2 > (C^2 - D^2)\sin^2(2\phi), \qquad \text{i.e.} \quad C \neq 0, \quad D \neq 0 \tag{3.34}$$

is satisfied. Under this assumption the full diagonalisation scheme works through and the main terms can be calculated directly and give (3.29). For completeness we also give a unitary diagonaliser of the matrix (3.33), namely

$$\begin{aligned} M_2(\phi) &= \frac{1}{\sqrt{2D^2\sin^2(2\phi) + 2C^2\cos^2(2\phi) + 2C\cos 2\phi\sqrt{:}}} \\ &\qquad \times \begin{pmatrix} C\cos 2\phi + \sqrt{:} & -D\sin 2\phi \\ D\sin 2\phi & C\cos 2\phi + \sqrt{:} \end{pmatrix} \\ &= \begin{pmatrix} m_1(\phi) & m_2(\phi) \\ -m_2(\phi) & m_1(\phi) \end{pmatrix} \end{aligned} \tag{3.35}$$

where $\sqrt{:} = \sqrt{C^2 - (C^2 - D^2)\sin^2(2\phi)}$, $\phi \neq \frac{\pi}{2}, \frac{3\pi}{2}$. Expressions are extended by continuity. ∎

3.3.1. *System form.* Again we use the diagonaliser $M(\epsilon, \phi)$ of $A(\epsilon, \phi)$ constructed in Proposition 3.6 to reformulate the thermo-elastic system as system of first order. Formulae (3.10) and (3.11) give the corresponding representation.

Remark 3.5. *1.* Since $M^{-1}(\epsilon,\phi)=\operatorname{diag}(1,M_2^*(\phi))(I-\epsilon N_1^{(1)}(\phi))+\mathcal{O}(\epsilon^2)$ in the notation of the proof of Proposition 3.6 it follows that the coupling functions satisfy

$$a_1(\epsilon,\phi)=1+\mathcal{O}(\epsilon^2) \tag{3.36a}$$

$$a_2(\epsilon,\phi)=\epsilon\frac{\tau-\lambda-2\mu}{\tau-\mu}(m_1(\phi)\cos\phi+m_2(\phi)\sin\phi)+\mathcal{O}(\epsilon^2) \tag{3.36b}$$

$$a_3(\epsilon,\phi)=\epsilon\frac{\tau-\lambda-2\mu}{\tau-\mu}(m_1(\phi)\sin\phi-m_2(\phi)\cos\phi)+\mathcal{O}(\epsilon^2) \tag{3.36c}$$

Since $\tau\neq\lambda+2\mu$ the function $a_2(\phi)$ vanishes only for $\phi=k\frac{\pi}{2}$, $k\in\mathbb{Z}$, while $a_3(\phi)$ vanishes for $\phi=(2k+1)\frac{\pi}{4}$, $k\in\mathbb{Z}$.
2. Note that in contrast to the conic situation the eigenvalues coincide to second order in the degenerate direction, while the coupling functions still vanish to first order (if we approach the degeneracy from parabolic directions).

3.3.2. *Asymptotic expansion of eigenvalues as $\epsilon\to 0$.* We write the coefficient matrix $B(|\xi|,\epsilon,\phi)$ as sum of homogeneous components in ϵ, cf. (3.17). This gives

$$B^{(0)}(|\xi|,\phi)=\begin{pmatrix}\sqrt{\tau}&&&&&&\mathrm{i}\gamma\\&\sqrt{\mu}&&&&&\\&&\sqrt{\mu}&&&&\\&&&-\sqrt{\tau}&&&\mathrm{i}\gamma\\&&&&-\sqrt{\mu}&&\\&&&&&-\sqrt{\mu}&\\-\frac{\mathrm{i}}{2}\gamma&&&-\frac{\mathrm{i}}{2}\gamma&&&\mathrm{i}\kappa|\xi|\end{pmatrix} \tag{3.37}$$

$$B^{(1)}(|\xi|,\phi)=\begin{pmatrix}&&&&&&0\\&&&&&&\mathrm{i}\gamma a_2^{(1)}(\phi)\\&&&&&&\mathrm{i}\gamma a_3^{(1)}(\phi)\\&&&&&&0\\&&&&&&\mathrm{i}\gamma a_2^{(1)}(\phi)\\&&&&&&\mathrm{i}\gamma a_3^{(1)}(\phi)\\0&\frac{\mathrm{i}}{2}\gamma a_2^{(1)}(\phi)&\frac{\mathrm{i}}{2}\gamma a_3^{(1)}(\phi)&0&\frac{\mathrm{i}}{2}\gamma a_2^{(1)}(\phi)&\frac{\mathrm{i}}{2}\gamma a_3^{(1)}(\phi)&0\end{pmatrix} \tag{3.38}$$

and $B^{(2)}(|\xi|,\phi)$ has entries on the diagonal, in the last row and last column. In order to apply a block-diagonalisation as $\epsilon\to 0$ we assume that the matrix $B^{(0)}(|\xi|,\phi)$ has as many distinct eigenvalues as possible. This is ensured if $\mu\neq\tau$, $\mu\neq\tau+\gamma^2$ and we can (1,2,1,2,1)-block-diagonalise this matrix family. Note, that due to the structure of the last rows and columns, the system decouples modulo ϵ^2 into a one-dimensional thermo-elastic system and one containing the elastic eigenvalues. The coupling comes only into play for the ϵ^3 entries.

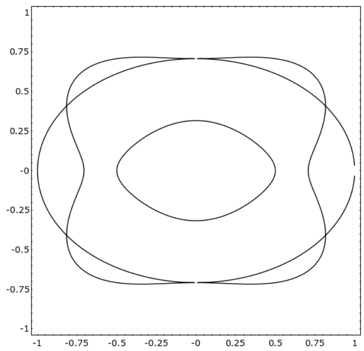

FIGURE 5. Cut of the Fresnel surface for hexagonal media, $\eta_2 = 0$. The parameters are chosen as $\tau_1 = 4$, $\tau_2 = 10$, $\lambda_1 = 2$, $\lambda_2 = 4$, $\mu = 2$. The complete surface is obtained by rotation along the vertical axis.

Proposition 3.7. *Assume $\mu \neq \tau$, $\mu \neq \tau + \gamma^2$. Then the eigenvalues and eigenprojections of $B(|\xi|, \epsilon, \phi)$ have full asymptotic expansions as $\epsilon \to 0$. The main terms are given by*

$$\nu_0(|\xi|, \epsilon, \phi) = \check{\nu}_0(|\xi|) + |\xi|\mathcal{O}(\epsilon^3), \tag{3.39a}$$

$$\nu_1^{\pm}(|\xi|, \epsilon, \phi) = \check{\nu}_1^{\pm}(|\xi|) + |\xi|\mathcal{O}(\epsilon^3), \tag{3.39b}$$

$$\nu_{2/3}^{\pm_1,\pm_2}(|\xi|, \epsilon, \phi) = \pm_1 \sqrt{\mu}|\xi| + \frac{C \pm_2 \sqrt{C^2 - (C^2 - D^2)\sin^2(2\phi)}}{4\sqrt{\mu}}|\xi|\epsilon^2 + |\xi|\mathcal{O}(\epsilon^3) \tag{3.39c}$$

where $\check{\nu}_0(|\xi|)$ and $\check{\nu}_1^{\pm}(|\xi|)$ are eigenvalues of the one-dimensional thermo-elastic system with parameter $\sqrt{\tau}$. The signs $\pm_1$ and $\pm_2$ are independent and the parameters C and D are given by (3.30).

Remark 3.6. Similar to Proposition 3.5 this statement is uniform in $|\xi|$. It will be of particular importance for us that the hyperbolic eigenvalues $\nu_{2/3}^{\pm}$ coincide up to second order in ϵ with the corresponding (roots of) eigenvalues of the elastic operator. This will be the key observation to transfer stationary phase estimates from elastic systems to the thermo-elastic one.

3.4. **Hexagonal media.** Finally we want to discuss the case of hexagonal media. The elastic operator defined by (1.4)–(1.5) is invariant under rotations in x_3-direction. We will make use of this fact and reduce considerations to a two-dimensional situation corresponding to cross-sections of the Fresnel surface.

As already pointed out in Section 2.5.4 degenerate directions are $\pm(0, 0, 1)^\top$, which are uniplanar. They could be handled similar to the cubic case, but rotational invariance makes estimates simpler. There are further circles of degenerate directions if

$$\tau_2 - 2\tau_1 \geq \lambda_2 + 2\mu. \tag{3.40}$$

We exploit rotational symmetry and consider the system only in the frequency hyperplane $\eta_1 = 0$. Then it is possible to express the eigenvectors $r_j(\eta)$ corresponding to eigenvalues $\varkappa_j(\eta)$ smoothly and the thermo-elastic system can be reformulated as system of first order in full analogy to the general treatment in Section 2. The derived

asymptotic expansions for eigenvalues and the description of their behaviour transfers away from the degeneracy and has to be equipped with an additional description near these degenerate directions.

Apart from the ξ_3-axis it is possible to find smooth families of eigenvectors $r_j(\eta)$ of $A(\eta)$. This follows directly from rotational invariance combined with one-dimensional perturbation theory of matrices, [6]. If we assume that the frequency support of initial data and therefore of the solution U, θ is conically separated from the uniplanar directions we can follow Section 2 and rewrite as first order system in $V(t,\xi)$ with coefficient matrix $B(\xi)$ given by (2.9) and of (1,1,5)-block structure. In what follows, we will ignore the scalar hyperbolic blocks and consider the remaining 5×5 matrix.

Based on the discussions from Section 2.5.4 we know that this 5×5 block is non-degenerate in the sense that its 1-homogeneous part has distinct eigenvalues if $(\tau_1 - \mu)(\tau_2-\mu)\neq 0$. We assume this in the sequel. But this means that the theory of Section 2 is applicable and gives a full discription of eigenvalues and eigenprojections and we are done.

Near the uniplanar directions, i.e., on the ξ_3-axis, we follow the same approach as for cubic media. We introduce polar co-ordinates around this direction and construct expressions for the corresponding asymptotics. There is one major simplification, due to rotational invariance the construction is independent of the angular variable.

4. Dispersive estimates

We will show how to use the information obtained in Sections 2 and 3 to derive L^p–L^q decay estimates for solutions to thermo-elastic systems. Some of the ideas we present are general in the sense that they can be applied to arbitrary space dimensions, however, our moin focus will be the three-dimensional case and the examples considered in Section 3.

The estimates we have in mind are micro-localised to (a) non-degenerate parabolic, (b) non-degenerate hyperbolic or (c) degenerate directions. The first two situations generalise the consideration of [16], [22] taking also into account the estimates due to Sugimoto [19], [20], while the treatment of degenerate directions is inspired by the work of Liess [7], [8], [10].

4.1. **Non-degenerate directions.**

We will consider two situations and micro-localise solutions to either open sets of parabolic directions or tubular neighbourhoods of compact parts of regular submanifolds of hyperbolic directions.

4.1.1. *Estimates in parabolic directions and for parabolic modes.* Let first $\psi,\tilde\psi\in C_0^\infty(\mathbb{S}^{n-1})$ be supported in $\mathcal{U}$ with $\tilde\psi=1$ on $\operatorname{supp}\psi$ and $\chi\in C^\infty(\mathbb{R}_+)$ a cut-off function satisfying $\chi(s)=0$ for $s\le\epsilon$ and $\chi(s)=1$ for $s\ge 2\epsilon$. We extend both ψ and $\tilde\psi$ as 0-homogeneous functions to $\mathbb{R}^n$. Then we consider the solution to the first order system

$$\mathrm{D}_tV=B(D)V,\qquad V(0,\cdot)=\tilde\psi(D)V_0, \tag{4.1}$$

with data $V_0\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^{2n+1})$. Note, that this is well-defined and $B(\xi)$ needs only to be defined on $\operatorname{supp}\tilde\psi$.

Lemma 4.1 (Parabolic estimate)**.** *Assume that* $\operatorname{supp}\psi$ *is contained in the set of parabolic directions. Then the solutions to* (4.1) *satisfy the a-priori estimates*

$$\|\chi(|D|)\psi(D)V(t,\cdot)\|_q \lesssim \mathrm{e}^{-Ct}\|V_0\|_{p,r}, \tag{4.2a}$$

$$\|(1-\chi(|D|))\psi(D)V(t,\cdot)\|_q \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|V_0\|_p \tag{4.2b}$$

for all $1 \le p \le 2 \le q \le \infty$ *and with Sobolev regularity* $r > n(1/p - 1/q)$.

Sketch of proof. The proof of this estimate is straightforward from the two-dimensional situation considered in [16]. For small frequencies we write the solution V as sum

$$V(t,x) = \sum_{\nu(\xi)\in\operatorname{spec}B(\xi)} \mathrm{e}^{\mathrm{i}t\nu(\mathrm{D})}P_\nu(\mathrm{D})V_0, \tag{4.3}$$

P_ν the corresponding eigenprojections. We know that $\|P_\nu(\xi)\| \lesssim 1$ and $\operatorname{Im}\nu(\xi) \sim |\xi|^2$ by Proposition 2.5. Now each of the appearing terms can be estimated using the L^p–$L^{p'}$ boundedness of the Fourier transform (for $p \in [1,2]$) and Hölder inequality. Similarly, the representation (4.3) in combination with the bound $\operatorname{Im}\operatorname{spec}B(\xi) \ge C$ gives exponential decay of L^2 and H^s norms and this combined with Sobolev embedding yields the desired estimate.

For intermediate frequencies we may have to deal with multiplicities and resulting singularities of the spectral projections. Instead of (4.3) we use a spectral calculus representation which implies

$$|\hat{V}(t,\xi)| \le \mathrm{e}^{-Ct}\frac{1}{2\pi}\int_\Gamma \|(\zeta - B(\xi))^{-1}\|\mathrm{d}\zeta \lesssim \mathrm{e}^{-Ct} \tag{4.4}$$

based on the compactness of the relevant set of frequencies ξ and the bound on the imaginary part due to Corollary 2.8 / Proposition 2.9. Here, Γ is a smooth curve encircling the family of parabolic eigenvalues for the relevant ξ. $\square$

If we consider hyperbolic directions we know that the parabolic eigenvalues are separated from the hyperbolic ones and we can use the spectral projection associated to the group of parabolic eigenvalues to separate them from the hyperbolic one(s). In this case the estimate of the above theorem is valid for the corresponding 'parabolic modes' of the solution. So we can restrict consideration to hyperbolic eigenvalues near hyperbolic directions.

4.1.2. *Treatment of non-degenerate hyperbolic directions.* We consider *only* the for us interesting case when hyperbolic directions form part of a regular submanifold of $\mathbb{S}^{n-1}$ and coupling functions vanish to first order, i.e., we assume that the corresponding coupling function $a_j : \mathbb{S}^{n-1} \supset \mathcal{U} \to \mathbb{R}$ satisfies

$$\mathrm{d}a_j(\eta) \ne 0 \quad \text{when } a_j(\eta) = 0,\ \eta \in \mathcal{U}. \tag{4.5}$$

This implies that $M_j = \{\eta \in \mathcal{U} \ :\ a_j(\eta) = 0\}$ is regular of dimension $n-2$, the normal derivative $\partial_n a_j(\eta) \ne 0$ never vanishes and $a_j(\eta) \le \epsilon$ defines a tubular neighbourhood of M_j with a natural parameterisation. The desired dispersive estimate is related to geometric properties of the section $\mathcal{S}_{(M_j)}$ of the Fresnel surface lying directly over M_j,

$$\mathcal{S}_{(M_j)} = \{\omega_j^{-1}(\eta)\eta : \eta \in M_j\} = \mathcal{S}_j \cap \operatorname{co} M_j. \tag{4.6}$$

Here $\operatorname{co} M_j$ denotes the cone over M_j. For dimensions $n \geq 4$ we have to distinguish between different cases, depending on whether the cross-section $\mathcal{S}_{(M_j)}$ of the Fresnel surface satisfies a convexity assumption or not. By the latter we mean that any intersection of $\mathcal{S}_j$ with a hyperplane tangent to $\operatorname{co} M_j$ is convex in a neighbourhood of $\mathcal{S}_{(M_j)}$.

If this convexity assumption is satisfied (or if $n = 3$ and therefore $\dim M_j = 1$), we define the convex Sugimoto index of $\mathcal{S}_{(M_j)}$ as maximal order of contact of $\mathcal{S}_{(M_j)}$ with hyperplanes normal to $\operatorname{co} M_j$.

Theorem 4.2 (Hyperbolic estimate, convex case)**.** *Assume that ψ is supported in a sufficiently small tubular neighbourhood of the regular hyperbolic submanifold M_j and that $\mathcal{S}_{(M_j)}$ satisfies the convexity assumption. Let further $\gamma_j = \gamma(\mathcal{S}_{(M_j)})$ be defined as above.*

Then the solutions to (4.1) *satisfy the a-priori estimate*

$$\|\psi(\mathrm{D}) P_{\nu_j}(\mathrm{D}) V(t,\cdot)\|_q \lesssim (1+t)^{-(\frac12 + \frac{n-2}{\gamma_j})(\frac1p - \frac1q)} \|V_0\|_{p,r} \tag{4.7}$$

for all $p \in (1,2]$, $pq = p+q$ and with Sobolev regularity $r > n(1/p - 1/q)$.

Proof. First, we outline the strategy of the proof. We split variables in the tubular neighbourhood of the regular hyperbolic submanifold M_j, one coordinate being the defining function $a_j(\eta)$ and the other parameterising points on M_j. We have to combine a (simple) parabolic type estimate in normal directions taking care of the imaginary part of the phase with stationary phase estimates for the integration along M_j. The stationary phase estimate is done first and follows the lines of [19], [20] along with [18, Sect. 5].

It is sufficient to show the estimate for $t \geq 1$. We follow the treatment of Brenner [2] and decompose the Fourier integral representing the corresponding hyperbolic modes of the solution V into dyadic pieces. For large and intermediate frequencies this amounts to estimate for all $k \in \mathbb{N}_0$

$$\mathcal{I}_k(t) = \sup_{z \in \mathbb{R}^n} \bigg| \int_{\tilde\eta=-\epsilon}^{\tilde\eta=\epsilon} \int_{\check\eta \in M_j} \int_{2^{k-1} \leq |\xi| \leq 2^{k+1}} \mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta + |\xi|^{-1}\nu_j(\xi))} \times p_j(\xi) \chi_k(\xi) |\xi|^{n-1-r} \mathrm{d}|\xi| \mathrm{d}\check\eta \mathrm{d}\tilde\eta \bigg| \tag{4.8}$$

with the notation $z = x/t$, $\xi = |\xi|\eta$, $\eta \simeq (\check\eta, \tilde\eta)$ with $\check\eta \in M_j$ and $\tilde\eta = a_j(\eta)$. The amplitude $p_j(\xi)$ arises from the spectral projector $P_{\nu_j}(\mathrm{D})$ and the phase $\nu_j(\xi)$ is complex-valued with $\operatorname{Im} \nu_j(\xi) \sim \tilde\eta^2$ uniform in $\xi \in \operatorname{supp} \chi_k$ and $k \in \mathbb{N}_0$.

If $z + \nabla_\xi \nu_j(\xi) \neq 0$, $\xi/|\xi| \in M_j$ or if z is not near a direction from M_j, the principle of non-stationary phase implies and gives a rapid decay. It suffices to restrict to z corresponding to stationary points. We use the method of stationary phase to estimate the integral over M_j, this can be done uniformly over ξ and $\tilde\eta$, provided ϵ is chosen small enough and yields an estimate of the form

$$\left| \int_{\check\eta \in M_j} \dots \mathrm{d}\check\eta \right| \leq C t^{-\frac{n-2}{\gamma_j}} |\xi|^{n-1-r-\frac{n-2}{\gamma_j}} \mathrm{e}^{-c\tilde\eta^2 t} \tag{4.9}$$

uniform in k and $|\tilde\eta| \leq \epsilon$. In order to obtain this estimate we apply Ruzhansky's multidimensional van der Corput lemma, [17], based on the uniformity of the Sugimoto index

$\gamma(\mathcal{S}_j \cap \operatorname{co}\{\eta : a_j(\eta) = \tilde\eta,\ \eta \approx \check\eta\})$ for small $\tilde\eta$ and the uniform bounds on the appearing amplitude. Similar to [16] the imaginary part of the phase can be incorporated in the estimate for the amplitude. Integration over $\tilde\eta$ yields a further decay of $t^{-1/2}$, while integrating over ξ and using $|\xi| \sim 2^k$ yields

$$\mathcal{I}_k(t) \le Ct^{-\frac12-\frac{n-2}{\gamma_j}} 2^{k(n-r-\frac{n-2}{\gamma_j})}. \tag{4.10}$$

Hence, we need $r \ge n - \frac{n-2}{\gamma_j}$ (compared to the elasticity or wave equation with $r \ge n - \frac{n-1}{\gamma}$) to apply Brenner's argument and obtain the desired estimate for the high frequency part. The required regularity follows from using Sobolev embedding for small t.

The treatment of small frequencies is somewhat simpler. We do not apply a dyadic decomposition, but still have to use a stationary phase argument along M_j combined with the behaviour of the imaginary part of the phase away from it,

$$\begin{aligned}\mathcal{I}(t) &= \sup_{z\in\mathbb{R}^n}\left|\int_{\tilde\eta=-\epsilon}^{\tilde\eta=\epsilon}\int_{\check\eta\in M_j}\int_{|\xi|\le1} \mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta+|\xi|^{-1}\nu_j(\xi))} p_j(\xi)\chi(\xi)|\xi|^{n-1}\mathrm{d}|\xi|\mathrm{d}\check\eta\mathrm{d}\tilde\eta\right|\\ &\le Ct^{-\frac{n-2}{\gamma_j}}\int_{|\xi|\le1}\int_{\tilde\eta=-\epsilon}^{\tilde\eta=\epsilon}\mathrm{e}^{-c\tilde\eta^2t|\xi|^2}|\xi|\mathrm{d}\tilde\eta|\xi|^{n-2-\frac{n-2}{\gamma_j}}\mathrm{d}|\xi| \le Ct^{-\frac12-\frac{n-2}{\gamma_j}}.\end{aligned}$$

□

Without proof we comment on the non-convex situation. If the convexity assumption is violated we have to replace the convex Sugimoto index by a corresponding non-convex one $\gamma_0(\mathcal{S}_{(M_j)})$. This is defined as the maximum over the minimal contact orders of $\mathcal{S}_{(M_j)}$ with hyperplanes normal to the cone $\operatorname{co} M_j$, the maximum taken over all points of $\mathcal{S}_{(M_j)}$. The price we have to pay for non-convexity is a loss of decay.

Theorem 4.3 (Hyperbolic estimate, non-convex case)**.** *Assume that ψ is supported in a sufficiently small tubular neighbourhood of the regular hyperbolic submanifold M_j and that $\mathcal{S}_{(M_j)}$ does not satisfy the convexity assumption. Let further $\widetilde{\gamma_j} = \gamma_0(\mathcal{S}_{(M_j)})$ be the non-convex Sugimoto index.*

Then the solutions to (4.1) *satisfy the a-priori estimate*

$$\|\psi(\mathrm{D})P_{\nu_j}(\mathrm{D})V(t,\cdot)\|_q \lesssim (1+t)^{-(\frac12+\frac{1}{\widetilde{\gamma_j}})(\frac1p-\frac1q)}\|V_0\|_{p,r} \tag{4.11}$$

for all $p \in (1,2]$, $pq = p+q$ and with Sobolev regularity $r > n(1/p-1/q)$.

4.1.3. *Application to cubic and hexagonal media.* Because of its importance later on we remark that in our applications to three-dimensional thermo-elasticity the manifolds M_j are parts of circles on $\mathbb{S}^2$, i.e. can be seen as intersections of $\mathbb{S}^2$ with a cone. So we have to look at the corresponding sections of the Fresnel surface. In this case γ_j is just the maximal order of tangency between the curve $\mathcal{S}_{(M_j)}$and its tangent lines. If the curvature of this curve is nowhere vanishing, then $\gamma_j = 2$. Furthermore, algebraicity of $\mathcal{S}$ of order 6 implies that the highest order of contact is 6 and therefore $\gamma_j \in \{2,\dots,6\}$ is the admissible range of these indices.

For cubic media there are two types of regular hyperbolic submanifolds. One is up to symmetry given by the circle $\eta_3 = 0$ on $\mathbb{S}^2$ and the corresponding eigenvalue is

equal to μ. Thus the section of the Fresnel surface is just a circle and therefore its curvature is nowhere vanishing. Similarly, for intersections of the Fresnel surface with the plane $\eta_2 = \eta_3$ we obtain the hyperbolic eigenvalue $\varkappa = \eta_2^2(\tau-\lambda)+\eta_1^2\mu$. It is a simple calculation[3] to show that the curvature of the corresponding section of the Fresnel surface is nowhere vanishing as soon as $\lambda \neq \tau$ and $\mu \neq 0$. Hence, $\gamma_j = 2$ in both cases.

For hexagonal media regular hyperbolic submanifolds correspond to circles on the Fresnel surface. Again, $\gamma_j = 2$.

4.2. **Cubic media in 3D.** We want to discuss the derivation for estimates near degenerate directions by the example of cubic media in three-dimensional space and combine them with the general estimates from Section 4.1.

4.2.1. *Conic points.* The following statement resembles [8, Thm. 1.5]. In [10, Sect. 3] a stronger decay rate is obtained for some conic degenerations, but they require a sufficiently bent cone and we can not guarantee that in our case.

Theorem 4.4 (Conic degeneration)**.** *Assume U_1, U_2 and θ_0 are micro-locally supported in a sufficiently small conical neighbourhood of a conically degenerate point on $\widehat{\mathbb{S}^2}$. Then the corresponding solution to the thermo-elastic system for cubic media satisfies the a-priori estimate*

$$\|\sqrt{A(\mathrm{D})}U(t,\cdot),U_t(t,\cdot),\theta(t,\cdot)\|_q \lesssim (1+t)^{-\frac12(\frac1p-\frac1q)}\|\sqrt{A(\mathrm{D})}U_1,U_2,\theta_0\|_{p,r} \tag{4.12}$$

for $p\in(1,2]$, $pq=p+q$ and $r>3(1/p-1/q)$.

Proof. The main idea is that the proof of [7] uses polar co-ordinates around the singularities of the Fresnel surface similar to our treatment in Section 3. Stationary phase arguments are applied in tangential direction and are uniform for small radii, while the final estimate follows after integration over the remaining variables.

It suffices to prove the statement for $t\ge 1$, the small time estimate is a direct consequence of Sobolev embedding theorem in combination with the obvious energy estimate. Similar to the hyperbolic estimate discussed before, we apply a dyadic decomposition of frequency space (localised to a small conic neighbourhood of the degenerate direction). The estimate for single dyadic components follows [7] resp. [8, Thm. 1.5]; the only thing we have to check is that the necessary assumptions are satisfied uniform with respect to $|\xi|$ and $k\in\mathbb{N}$. We consider

$$\begin{aligned}\mathcal{I}_k(t) = \sup_{z\in\mathbb{R}^3}\bigg|\int_0^{\tilde\epsilon}\int_0^{2\pi}\int_{2^{k-1}\le|\xi|\le 2^{k+1}} &\mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta+|\xi|^{-1}\nu_j(|\xi|,\epsilon,\phi))}\\ &\times\ p_j(|\xi|,\epsilon,\phi)\chi_k(|\xi|)|\xi|^{2-r}\mathrm{d}|\xi|\mathrm{d}\phi\epsilon\mathrm{d}\epsilon\bigg|,\end{aligned} \tag{4.13}$$

where $\eta\in\mathbb{S}^2$ denotes the point with polar co-ordinates (ϵ,ϕ) near the conic degenerate direction and $\xi=|\xi|\eta$. The amplitude $p_j(|\xi|,\epsilon,\phi)$ arises from the spectral projector (given in terms of the diagonaliser) constructed in the blown-up polar co-ordinates and

[3]Parametrising by the angle, the hyperbolic eigenvalue is given by $\varkappa(\phi)=\mu+\frac{\tau-\lambda-2\mu}{2}\sin^2\phi$ and it remains to check that $\partial_\phi^2\sqrt{\varkappa(\phi)}+\sqrt{\varkappa(\phi)}\neq 0$, see [22] for such a calculation.

$\chi_k(\xi)$ corresponds to the dyadic decomposition. The complex phase $\nu_j(|\xi|,\epsilon,\phi)$ is described in Proposition 3.5. Its imaginary part is non-negative and vanishes to second order in $\epsilon=0$ as well as for three hyperbolic manifolds emanating from the conic degenerate point. Again we may treat this imaginary part as part of the amplitude and apply stationary phase estimates for the integral with respect to ϕ. As the approximation of the phase modulo $\mathcal{O}(\epsilon^2)$ is independent of ϕ and uniform in $|\xi|$ this yields

$$\left|\int_0^{2\pi}\dots\mathrm{d}\phi\right|\lesssim t^{-\frac12}|\xi|^{\frac32-r}\epsilon^{\frac12} \tag{4.14}$$

uniform in $|\xi|$, k and $0\le\epsilon\le\tilde\epsilon$. There is no further benefit from the imaginary part (as there can not be a lower bound with respect to ϵ) and integrating with respect to $|\xi|$ and ϵ concludes the estimate for $\mathcal{I}_k(t)$. Similarly, we estimate the small frequency part

$$\begin{aligned}\mathcal{I}(t)&=\sup_{z\in\mathbb{R}^3}\left|\int_0^{\tilde\epsilon}\int_0^{2\pi}\int_{|\xi|\le1}\mathrm{e}^{\mathrm{i}t|\xi|(z\cdot\eta+|\xi|^{-1}\nu_j(|\xi|,\epsilon,\phi))}p_j(|\xi|,\epsilon,\phi)\chi(|\xi|)|\xi|^2\mathrm{d}|\xi|\mathrm{d}\phi\epsilon\mathrm{d}\epsilon\right|\\&\le Ct^{-\frac12},\end{aligned} \tag{4.15}$$

such that Brenner's method again yields the desired decay estimate. □

4.2.2. *Uniplanar points.* The treatment of uniplanar degeneracies follows [8]. We have to make one further additional assumption related to the shape of certain curves on the Fresnel surface near the degenerate point. To be precise, we either require that

$$\Omega\cap\mathcal{S}\cap\Pi\quad\text{has non-vanishing curvature} \tag{4.16}$$

for $\Omega\subset\mathbb{R}^n$ an open neighbourhood of the uniplanarly degenerate point and for any plane Π sufficiently close and parallel to the common tangent plane at the unode. This condition is equivalent to the technical assumption (1.12) made in [10] If (4.16) is violated, we need to consider Sugimoto indices $\gamma_u=\gamma(\Omega\cap\mathcal{S}\cap\Pi\subset\Pi)$, i.e., contact orders of these planar curves with its tangent planes combined with a uniformity assumption. Under assumption (4.16) the index is given by $\gamma_u=2$.

For cubic media we have to use the statement of Proposition 3.6 to determine the index γ_u. Using the notation of (3.30), it suffices to calculate the indices of the indicator curves determined by $\epsilon^2\big(\mu+C\pm\sqrt{C^2\cos^2(2\phi)+D^2\sin^2(2\phi)}\big)=1$. This yields

$$\gamma_u\in\{2,3,4\} \tag{4.17}$$

In the nearly isotropic case we have $\gamma_u=2$, away from it $\gamma_u=3$. Both are generic, while the borderline case with $\gamma_u=4$ is not. The asymptotic construction of the eigenvalues and eigenprojections near the uniplanarly degenerate point of Proposition 3.7 yields that the assumption is satisfied uniformly for the phase functions appearing in all dyadic components of the operator.

Theorem 4.5 (Uniplanar degeneration)**.** *Assume U_1, U_2 and θ_0 are micro-locally supported in a sufficiently small conical neighbourhood of a uniplanarly degenerate point on $\widehat{\mathbb{S}^2}$. Let further γ_u be the index of the uniplanar point. Then the corresponding solution to the thermo-elastic system for cubic media satisfies the a-priori estimate*

$$\|\sqrt{A(\mathrm{D})}U(t,\cdot),U_t(t,\cdot),\theta(t,\cdot)\|_q\lesssim(1+t)^{-(\frac12+\frac{1}{\gamma_u})(\frac1p-\frac1q)}\|\sqrt{A(\mathrm{D})}U_1,U_2,\theta_0\|_{p,r} \tag{4.18}$$

for $p\in(1,2]$, $pq=p+q$ and $r>3(1/p-1/q)$.

Sketch of proof. We will sketch the major differences to the treatment of conic degeneracies. We will again use polar co-ordinates and estimate corresponding dyadic components (4.13), where now $\nu_j(|\xi|,\epsilon,\phi)$ is determined by Proposition 3.7. The imaginary part of $\nu_j(|\xi|,\epsilon,\phi)$ vanishes to third order and is of no benefit, while the real part coincides to third order with the corresponding elastic eigenvalue. This allows to use estimates from [8] and [10, Sect. 4], the main difference to the previous situation is that we now use stationary phase estimates for both, the angular and the radial integral. The proof itself then coincides with the corresponding proof for cubic elasticity, cf. [11].

Using a change of variables the integral is written in the new variables $\omega_j(\eta)|\xi|$ (i.e., roughly $\operatorname{Re}\nu_j$) and $\eta/\omega_j(\eta)\in\mathcal{S}_j$. In this form the phase splits and the crucial estimate is just a Fourier transform of a density carried by the sheet of the Fresnel surface (with possible singularity in the unode). This is calculated by introducing *distorted* polar co-ordinates on the surface. As level sets we use cuts of the surface by planes parallel to the common tangent plane. Then we will at first apply the method of stationary phase to the radial variable in these co-ordinates. These stationary points are non-degenerate and we use the obtained first terms in the asymptotics for a second stationary phase argument in the angular variables. The condition (4.16) would imply again that stationary points are non-degenerate and we are done, while if (4.16) is violated we use the lemma of van der Corput instead to prove the estimate. □

4.2.3. *Collecting the estimates.* It remains to collect all the estimates into a final statement for cubic media. Parabolic directions are treated by Lemma 4.1; hyperbolic manifolds away from degenerate points are covered by Theorem 4.2. The remaining 24 degenerate directions fall into either of the previously discussed categories and estimates follow from Theorem 4.4 and 4.5. The resulting estimates are collected in Table 1.

	small frequencies	large frequencies
parabolic directions	$(1+t)^{-\frac{3}{2}}$	e^{-Ct}
hyperbolic directions	$(1+t)^{-1}$	$(1+t)^{-1}$
conic degeneracies	$(1+t)^{-\frac{1}{2}}$	$(1+t)^{-\frac{1}{2}}$
uniplanar degeneracies	$(1+t)^{-\frac{1}{2}-\frac{1}{\gamma}}$ $\gamma\in\{2,3,4\}$	$(1+t)^{-\frac{1}{2}-\frac{1}{\gamma}}$ $\gamma\in\{2,3,4\}$

Table 1. Contributions to the dispersive decay rate for cubic media.

Corollary 4.6 (Cubic decay rates)**.** *Cubic media in three space dimensions satisfy the dispersive type estimate*

$$\|\sqrt{A(\mathrm{D})}U(t,\cdot), U_t(t,\cdot), \theta(t,\cdot)\|_{L^q(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}\|\sqrt{A(\mathrm{D})}U_1, U_2, \theta_0\|_{p,r} \qquad (4.19)$$

for all data $U_1 \in W^{p,r+1}(\mathbb{R}^3;\mathbb{C}^3)$*,* $U_2 \in W^{p,r}(\mathbb{R}^3;\mathbb{C}^3)$ *and* $\theta \in W^{p,r}(\mathbb{R}^3)$*, provided* $p \in (1,2]$*,* $pq = p+q$ *and* $r > 3(1/p - 1/q)$*.*

Decay rates improve if the Fourier transform of the initial data vanishes in the conically degenerate directions. This could be achieved by posing particular symmetry conditions.

4.3. **Hexagonal media.** The treatment of hexagonal media is somewhat simpler. The uniplanar degenerations trivially satisfy the assumption (4.16) and therefore yield the decay rates specified by the above theorem. The additionally appearing manifolds of degenerate directions are trivially resolved as there are smooth families of eigenprojections associated to both eigenvalues (as we stay away from the uniplanar points) and we can therefore treat the modes separately.

One of them is hyperbolic for all directions, we refer to it as the genuine hyperbolic mode. The sheet of the Fresnel surface corresponding to this mode, i.e., to the eigenvalue $\varkappa(\eta) = \frac{\tau_1-\lambda_1}{2}(\eta_1^2+\eta_2^2)+\mu\eta_3^2$ is easily seen to be strictly convex for all choices of the parameter and gives therefore t^{-1}. The proof is similar to that for the wave equation, see [2].

The parabolic modes away from the degenerate hyperbolic directions are treated as before, while the remaining degenerate hyperbolic manifold is treated by the estimate of Theorem 4.2 with $\gamma = 2$ due to rotational invariance. The resulting estimates are collected in Table 2.

	small frequencies	large frequencies
genuine hyperbolic mode	$(1+t)^{-1}$	$(1+t)^{-1}$
parabolic modes	$(1+t)^{-3/2}$	e^{-Ct}
hyperbolic directions	$(1+t)^{-1}$	$(1+t)^{-1}$
uniplanar degeneracies	$(1+t)^{-1}$	$(1+t)^{-1}$

Table 2. Contributions to the dispersive decay rate for hexagonal media.

Corollary 4.7 (Hexagonal decay rates)**.** *Cubic media in three space dimensions satisfy the dispersive type estimate*

$$\|\sqrt{A(\mathrm{D})}U(t,\cdot),U_t(t,\cdot),\theta(t,\cdot)\|_{L^q(\mathbb{R}^n)} \lesssim (1+t)^{-(\frac{1}{p}-\frac{1}{q})}\|\sqrt{A(\mathrm{D})}U_1,U_2,\theta_0\|_{p,r} \qquad (4.20)$$

for all data $U_1 \in W^{p,r+1}(\mathbb{R}^3;\mathbb{C}^3)$*,* $U_2 \in W^{p,r}(\mathbb{R}^3;\mathbb{C}^3)$ *and* $\theta \in W^{p,r}(\mathbb{R}^3)$*, provided* $p \in (1,2]$*,* $pq = p+q$ *and* $r > 3(1/p-1/q)$*.*

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